The Risk-Return Frontier

In this lecture we begin to develop the theory of portfolio management.

Consider two assets X and Y which have associated random returns, r_x and r_y. A wealth portfolio can be constructed by holding these two assets. The proportion of wealth held in X can be labeled α and proportion in Y, is then $(1-\alpha)$. Let's label $(1-\alpha)$ as β for the moment for ease of derivation. The portfolio return is, then,

$$r_p = \alpha\, r_x + \beta\, r_y, \qquad 1$$

Its expected value is

$$E(r_p) = E(\alpha\, r_x + \beta\, r_y) = \alpha E(r_x) + \beta E(r_y) \qquad 2$$

The portfolio variance is somewhat different.

$$V(r_p) = V(\alpha\, r_x + \beta\, r_y)$$

$$= E[\alpha\, r_x + \beta\, r_y - E(\alpha\, r_x + \beta\, r_y)]^2$$

$$= E[\alpha r_x - E(\alpha r_x)]^2 + E[\beta r_y - E(\beta r_y)]^2 + 2\, E[(\alpha r_x - E(\alpha r_x))(\beta r_y - E(\beta r_y))]$$

$$= \alpha^2\, V(r_x) + \beta^2\, V(r_y) + 2\, \alpha\beta\, Cov(r_x, r_y) \qquad 3$$

where $Cov(r_x, r_y)$ is the covariance between the return to asset X and to asset Y. Covariance is defined in similar fashion to variance using the expected value operator. In practice, covariance is most easily thought of in terms of correlation:

$$Cov(r_x, r_y) = \rho_{xy}\sigma_x\sigma_y$$

where ρ is correlation and σ_i is the standard deviation of the return on asset i. It is useful to simplify the notation henceforward by substituting σ^2, σ_{ij}, and σ for variance, covariance, and standard deviation, respectively.

What we have, then, is expected portfolio return and portfolio risk. Portfolio return is the simple, linear mapping between the returns to the two assets as the portfolio composition changes. As α varies between zero and one, the portfolio varies from no X and all Y to all X and no Y. As α varies so too does the expected return for the portfolio as shown in eqt. (2).

Portfolio risk is the standard deviation of the portfolio return, which is the square root of the variance of the portfolio return. The portfolio risk is not a linear mapping like the expected portfolio return. That is, portfolio risk is not just the portfolio-share-weighted combination of the risk of the two assets. Portfolio risk is the square root of the variance of the portfolio return, which is the squared-share weighted sum of the variance of the returns on the two assets *plus* the covariance of their returns. This is what is shown in eqt. (3). These points are summarized below:

Expected Portfolio Return: $E(r_p) = \alpha E(r_x) + (1-\alpha)E(r_y)$ 4

Portfolio Risk $\sigma_p = [\alpha^2 \sigma_x^2 + (1-\alpha)^2 \sigma_y^2 + 2\, \alpha\, (1-\alpha)\, \rho_{xy}\sigma_x\sigma_y]^{1/2}$ 5

Together, the expected portfolio return and the portfolio risk create an Investment Frontier. In managing wealth, the individual moves along the frontier by varying the amount of wealth held in each asset. Equations (4) and (5) identify the investment possibilities that an individual faces if the individual's wealth is held in two assets. The equations are real. All the terms in the equations have empirical values.

For instance, if *X* is Ford Motor Company, we can estimate $E(r_x)$ by the average stock return that Ford has enjoyed over the last ten years. Let

$$r = \frac{P_1 + D}{P_0}$$

where P_0 is the stock price at the beginning of the year, P_1 is the stock price at the end of the year, and *D* are any dividends paid during the year. Similarly, σ_x can be estimated by the standard deviation of r_x over the ten years. If *Y* is Connecticut Water Service, Inc., the correlation between Ford and CWS can be estimated from the sample correlation of returns over the last ten years.

Equations (4) and (5) can be calculated. The question is, What does the Investment Frontier look like?

To answer this, first notice that portfolio risk is a linear mapping of the risks associated with the individual assets in the case where ρ is one. Where there is perfect correlation between the two assets, eqt. (5) becomes:

$$\sigma_p = [(\alpha\, \sigma_x + (1-\alpha)\, \sigma_y)\, (\alpha\, \sigma_x + (1-\alpha)\, \sigma_y)]^{1/2}$$

or

$$\sigma_p = \alpha\, \sigma_x + (1-\alpha)\, \sigma_y$$

In this case the Investment Frontier is simply a straight line connecting point *X* with point *Y*. This makes sense because the values of the two assets move together in lock step.

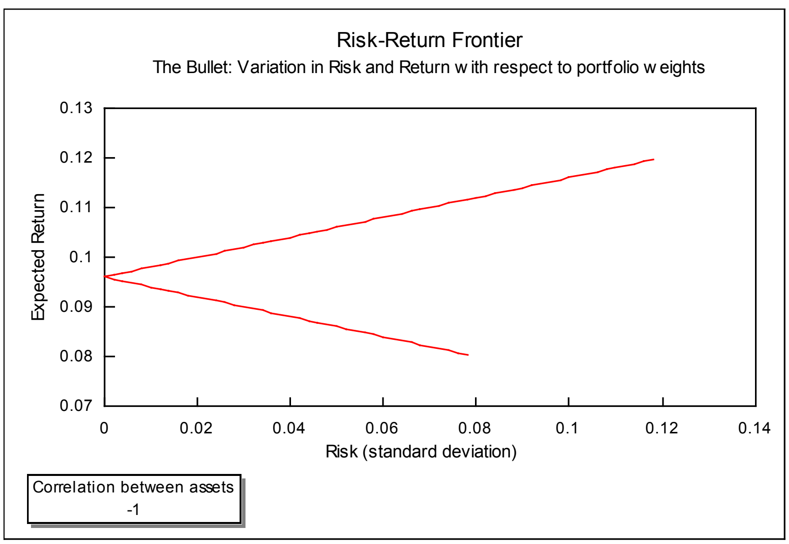
Next consider what happens when the correlation between the two assets is -1. In this case the Investment Frontier again becomes linear except that now the line connecting the risk-return for the two assets does not pass directly between them. The line from asset *X* (the higher risk-return asset) bounces off of the vertical axis at zero portfolio risk and then goes to the risk-return point associated with asset *Y*. This is shown in Figure 1 below.

When the correlation between the returns on two assets is -1, the assets are perfectly negatively related. When one goes up, the other goes down by exactly the same proportion. This special case is called a perfect hedge. By combining the appropriate amount of *X* and *Y* portfolio risk can be driven to zero.[1]

Now it is pretty clear what happens as the correlation between the returns to the two assets becomes less than perfect. For assets whose returns are less than perfectly correlated, the Investment Frontier lies inside the boundary formed by the perfectly positive and perfectly negative correlations. When the asset correlation is less than perfect, the portfolio risk deviates from linear mapping and the Investment Frontier becomes curved. In fact, because the curve, the Investment Frontier is commonly called the Bullet.

It is useful to consider the Investment Frontier graphically. Figures 2 & 3 show the risk-return tradeoff for correlations of .5 and .1. The end points of the frontier are the expected return and standard deviation of the two assets. As shown, we are considering an asset *Y* with an expected return of .08 and a risk of .08, while asset *X* has an expected return of .12 and a risk of .12. When the correlation between the two assets is .5, the Bullet has a slight, concave bow. As the correlation becomes weaker though still positive (.1), the Bullet takes a shape more fitting to its name.

Figure 1



[1] We usually say that *Y* is a perfect hedge for *X* because *Y* is the lower return asset. Note that even though the investor can construct a perfect hedge between these two assets, the investor will not necessarily exercise this in the construction of the optimal portfolio. The investor will choose the portfolio weights that allow the investor to reach the highest possible indifference curve in risk-return space.

Figure 2

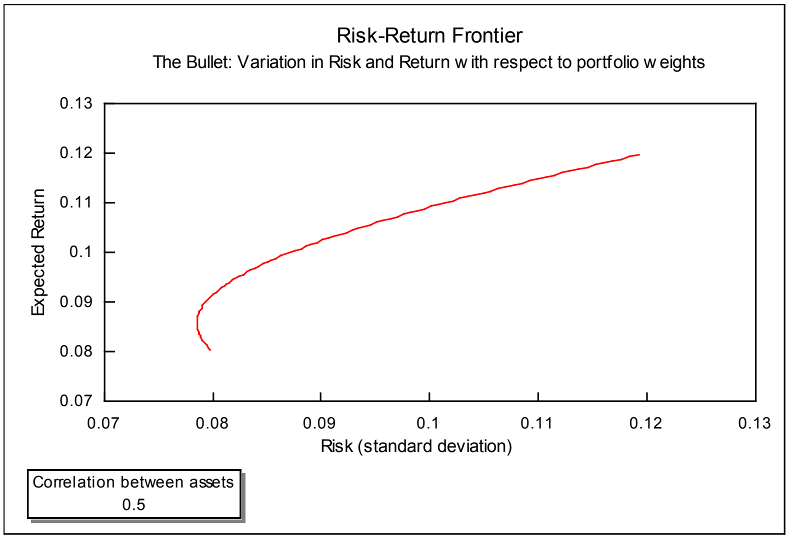
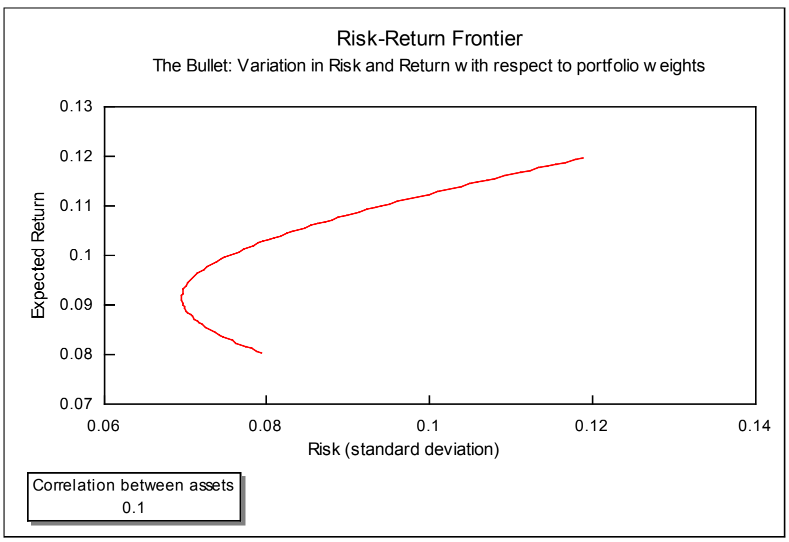


Figure 3



More on Diversification

The next step in our study of portfolio diversification is to consider what happens when we add a third asset to the mix. The expected portfolio return and standard deviation look like the following where w_i is the portfolio weight for the *i*th asset:

Expected Portfolio Return: $E(r_p) = w_1 E(r_x) + w_2 E(r_y) + w_3 E(r_z)$ 6

Portfolio Risk

$$\sigma_p = \left[w_1^2 \sigma_x^2 + w_2^2 \sigma_y^2 + w_3^2 \sigma_z^2 + 2 w_1 w_2 \sigma_x \sigma_y \rho_{xy} + 2 w_1 w_3 \sigma_x \sigma_z \rho_{xz} + 2 w_2 w_3 \sigma_y \sigma_z \rho_{yz} \right]^{-\frac{1}{2}} \qquad 7$$

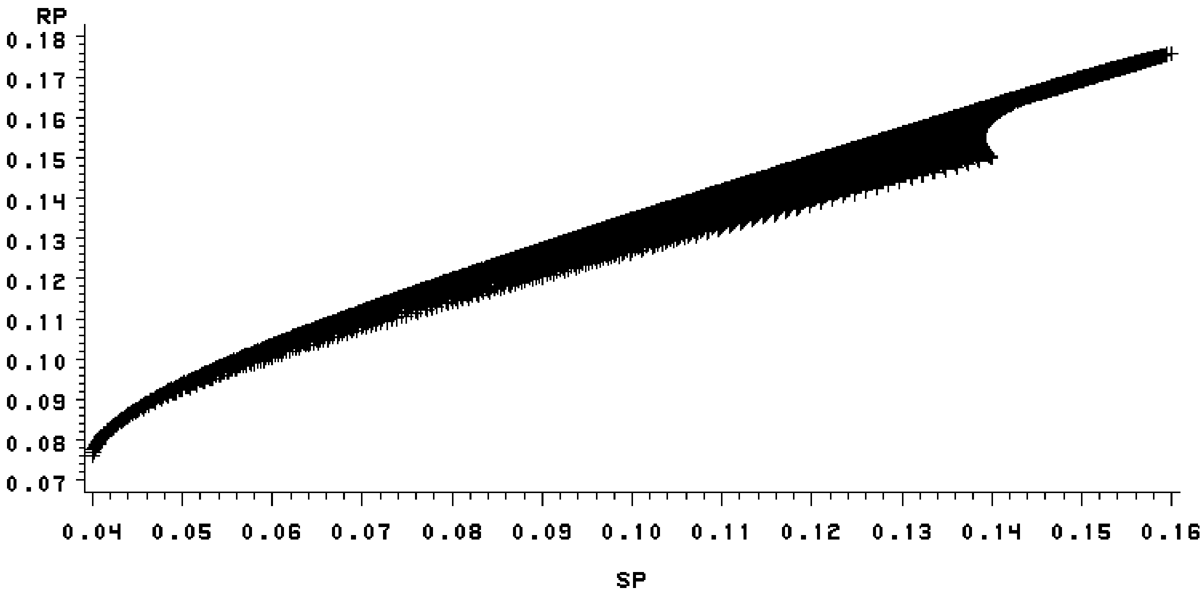
With three assets there are nine market parameters that we must know: the expected return on each asset, the variance of the return on each asset, and the pairwise correlations.

The following is a SAS program that computes all possible three asset portfolios and then plots the bullet:

```
data one;
rx=.176; ry=.15; rz=.076;
sx=.16; sy=.14; sz=.04;
vx=sx**2; vy=sy**2; vz=sz**2;
pxy=.79; pxz=.34; pyz=.3;

do a=0 to 1 by .01;
do b=0 to 1 by .01;
w1=a; w2=(1-a)*b; w3=(1-a)*(1-b);

rp=w1*rx+w2*ry+w3*rz;
sp=sqrt((w1**2)*vx + w2**2*vy + w3**2*vz
     +2*w1*w2*sx*sy*pxy + 2*w1*w3*sx*sz*pxz
      +2*w2*w3*sy*sz*pyz);
output;
end; end;
run;

proc gplot; plot rp*sp; run;
```

The market parameters are taken from the S&P 500 Index, the S&P Mid-Cap index, and the Fidelity Corporate Bond fund.

The plot of the bullet looks like this:



Notice that the lower contour represents the bullets of assets X and Y, and Y and Z. The portfolio choices mapping the envelope to the northwest are the efficient portfolios of three assets.

This picture shows the value of portfolio diversification and the gains from expanding the number of assets. The implication is that the individual always gains from adding an asset to the portfolio so long as the asset pushes the investment frontier towards the northwest.

First Principle of Asset Pricing

If an asset does not enhance the investment frontier by pushing it out toward the northwest, then that asset will be shunned by investors. This causes its price to fall. As its price falls, its return rises while its risk remains constant. At its return rises it will eventually enhance the investment portfolio.

Conclusion: all assets can be priced such that they each contribute to pushing the investment frontier to the northwest.

Second Principle of Asset Pricing

While the first principle of asset pricing offers an equilibrium pricing algorithm for a single investor, it must hold for all investors, that is, for investors who differ in their degree of risk aversion.

It is not obvious that it does. That is, if different investors choose portfolios along the concave investment frontier, they will value assets differently. Investors choosing portfolios more toward the southwest will value less risky assets more than investors choosing portfolios toward the northeast. The fact that investors have different asset valuations means that the prices of assets will change as investors change their relative consumption/investment decisions. As they trade assets to adjust consumption, the investment frontier changes. In this world, we cannot be sure that an equilibrium exists.

This problem is resolved when we introduce a *riskless* asset. Consider an asset that has zero risk, and hence, zero correlation with any point on the *risky* investment frontier. This zero risk asset means that all investors can satisfy their relative demands for risk aversion by choosing some of the riskless asset and some of the best risky portfolio. In this world, all investors choose the same risky portfolio. They all value each risky asset in the same way. Therefore, an equilibrium pricing of risky assets is assured.

The optimal risky portfolio is the one associated with a straight line drawn from the return on the riskless asset that is tangent to the investment frontier of risky assets. The riskless asset is uncorrelated with any point on the risky investment frontier. Hence, the investor can create portfolios by combining the riskless asset with any portfolio on the risky investment frontier. The investor will choose the risky portfolio that gives the highest risk/return tradeoff compared to the riskless asset. This is the portfolio associated with the tangent line of riskless asset to the risky investment frontier.

The implication is that there is a unique portfolio of risky assets. It is common to all investors. It changes as the return on the riskless asset changes. It also changes as the risk and return of any risky asset changes (because of exogeneous forces). However, the first principle of equilibrium pricing holds. Every risky asset must contribute to pushing the risky investment frontier to the northwest. Furthermore, the second principle holds because all investors seek to hold the same portfolio of risky assets. Hence all investors value all risky assets in the same way.