Capital Asset Pricing

The notion of a perfect hedge which was introduced in the last lecture is complicated but interesting. A simpler but similar concept is the possibility of a risk free return. Consider an asset *F* that has a return of r_f and has no risk. That is, the return on *F* occurs without variance; r_f is not a random variable; it has no standard deviation. In this case we can consider a portfolio that is composd of *F* and some risky asset *X*. The risk-return frontier in this case will be simply a straight line joining the risk-return combination of *X* with the return of *F* on the vertical axis. The slope of this line is just the rise over the run, or

$$\frac{E(r_x) - r_f}{\sigma_x} \tag{1}$$

Combining the risk-free asset with two risky assets gives us the bullet-shaped risky-asset investment frontier with a plank laid down on it stepping from the risk-free return to the highest point on the bullet. Thus, with two risky assets and one risk free, the consumer is able to enjoy the benefits of portfolio diversification along with the risk-reducing choices offered by the risk free asset.

Next, consider the investment risk-return frontier when there are numerous assets in addition to the risk free asset. With numerous risky assets available to the investor, portfolio wealth can be spread across any or all. In this case the expected portfolio return can be written as:

$$E(r_p) = \sum_{i=1}^{n} \omega_i E(r_i)$$

where ω_i are the weights assigned to each of the *n* risky assets that the investor chooses. If the investor chooses only risky assets the ω′s sum to one. Similarly, the variance of the portfolio return can be defined as:

$$V(r_p) = \sum_{i=1}^{n} \sum_{i=j}^{n} \omega_i \omega_j \sigma_{ij} \tag{2}$$

where s_{ii} is the same thing as the variance of the return on the *i*th asset, s_i^2.[1] The investor's risk-return frontier is still bullet shaped even though it is more difficult to depict the frontier in this case because of the multidimensional variation in the portfolio. The portfolio can vary by changing the weight assigned to each of *n* assets, and portfolio variance is more complicated because of multiple covariances. Consider the simple example of three assets, *X, Y,* and *Z.*

With three assets, one can imagine a bullet between two of the assets. Each point on the two asset bullet defines a relative portfolio weight for the first two assets. Let α_1 be the set of relative portfolio weights that generate an investment frontier between assets *X* and *Y*. The two asset bullet can then be merged with the third asset. For each of the relative portfolio weights allocating α_1 to *X* and $(1-\alpha_1)$ to *Y*, there are a set of weights between the asset *Z* and the portfolio of *X* and *Y*. Call these weights α_2. For instance, if α_1 is .5 and α_2 is .5, then 25% of the portfolio is in *X*, 25% in *Y*, and 50% in *Z*. The bullet associated with the three asset portfolio is the envelope of all the bullets that can be generated as all possible combinations of α_1 and α_2 are considered. By envelope, we mean that the bullet is the most efficient combinations of the three assets, that is, the highest return relative to each possible level of risk.

[1] See p. 173+ in Copeland and Weston for a good discussion of the algebra portfolio variance.

In like fashion, an investment frontier for all assets exists. As the portfolio is expanded in terms of the number of assets, from a subset *n* to all risky investment options *N*, a globally efficient envelope investment frontier results.

It is somewhat instructive to consider what happens to the variance of this global portfolio as the share in any asset changes. This can be found by defining portfolio variance as given in eqt. (2) over all assets *N* and then differentiating with respect to the share held in the *i*th asset:

$$\frac{\partial V(r_p)}{\partial \omega_i} = 2\omega_i\sigma_i^2 + 2\sum_{\substack{j=1 \\ j\neq i}}^{N} \omega_j \sigma_{ij} \qquad 3$$

Now consider what happens to portfolio variance as *N* goes to infinity. In this limit, the individual porfolio weights go to zero. This means that the first term on the right-hand side of eqt. (3) becomes infinitely small. The second term, however, is summed over the infinitely large *N* so it does not vanish. The implication is that the own variance of an asset becomes a trivial component of portfolio variance, while its covariance with other assets continues to be important.

Given the efficient frontier of investment options identified by the global envelop of all bullets, the investor can link this to the risk free rate. The risk free rate is a point on the vertical axis of the risk-return space. This point can be projected to a tangency with the efficient risk-return frontier. Where this tangency occurs defines the most efficient of the efficient portfolios. We call this the **market portfolio**. The return associated with this portfolio is the **market return.**

The line from the risk-free return through this point is called the **capital market line**. The capital market line is important because it identifies the most efficient investment options facing the consumer. By holding a combination of the market portfolio and the risk-free return the consumer can beat any other portfolio of risky assets. If the consumer wants less risk than the market, the consumer holds some risk-free asset and some market. If the consumer wants more risk than the market, the consumer can borrow at the risk-free rate and invest in the market. The consumer essentially separates the portfolio of risky assets from the riskless asset. The consumer maximizes utility by choosing the point on the capital market line where the highest possible indifference curve is tangent.

Asset Pricing and the Market Portfolio

The stage is now set to pull the rabbit out of the hat. We have identified consumer behavior in terms of utility maximization in the face of risk aversion. We have shown how portfolio diversification among risky assets offers superior risk-return tradeoffs compared to holding single risky assets. We have shown that given a risk-free asset, there is a unique portfolio of choice for risky assets. Now we show how assets are priced.

Simply enough, given that there is a unique portfolio of choice for risky assets, that is, given that there is one portfolio that everyone holds, each asset must be priced into that portfolio. The simplest way to think about this is to consider only two risky assets, *X* with the higher risk and return and *Y* with the lower. What if when the risk-free plank was laid down on the bullet of the two assets it hit only the higher return point? This would mean, of course, that investors would only be interested in holding the one asset. This would cause people to bid down the price of *Y* and bid up the price of *X*. However, these price changes would change the returns to the two assets. As the price of Y falls, its return increases; as the price of X increases, its return is driven down.[2] This changes the bullet and the market portfolio, but in the two asset case, we can imagine how the equilibrium develops. The same thing happens when there are many risky assets.[3]

Now that the rabbit is out of the hat, all we need do is show that it is a real bunny by letting it run around a bit. To do this it is enlightening to consider how market portfolios that are inefficient look when compared to the efficient one. We define a portfolio of two assets: One is the efficient, market portfolio treated as one asset. That is, let there be a mutual fund that is composed of all risky assets in their efficient weights. This mutual fund is available to investors to hold in any amount. The second asset we wish to consider is an individual risky asset, X_i, which has a return r_i. The expected portfolio return and portfolio risk are defined just as they were in our last lecture when we considered portfolios of X and Y. Here we consider what happens to the diversification frontier when the investor chooses to hold a portion of his wealth, ω, in asset i and the rest, (1-ω), in the market.[4]

Expected Portfolio Return: $$E(r_p) = \omega E(r_i) + (1-\omega)E(r_m) \qquad 4$$

Portfolio Risk $$s_p = [\omega^2 s_i^2 + (1-\omega)^2 s_m^2 + 2\,\omega\,(1-\omega)\,s_{im}]^{1/2} \qquad 5$$

We now do what we refrained from doing in the last lecture in order to heighten the dramatic effect. Let's differentiate both eqt's (4) and (5) with respect to ω. The derivative of eqt. (4) is simple.

$$\frac{\partial E(r_p)}{\partial \omega} = E(r_i) - E(r_m) \qquad 6$$

The derivative of portfolio risk is a little messy but it will simplify nicely in a moment:

$$\frac{\partial \sigma_p}{\partial \omega} = \frac{1}{2}\frac{[2\omega\sigma_i^2 - 2(1-\omega)\sigma_m^2 + 2\sigma_{im} - 4\omega\sigma_{im}]}{\sigma_p} \qquad 7$$

We wish to evaluate eqt. (7) at the point where ω is equal to zero. This requires a little explaining.

The market portfolio contains asset i in the efficient weight ω_i. The question posed by eqt. (7) is, what happens to portfolio risk when this asset is increased beyond its optimal amount? By evaluating eqt. (7) at $\omega = 0$ we can determine the effect on portfolio risk when this occurs. Remember that the market portfolio as a point on the efficient investment frontier. The market portfolio is arbitrarily chosen from the efficient frontier by its tangency with the line from the risk-free return. Every asset has an efficient weight in the market portfolio. This weight is different all along the efficient frontier. What the derivative in eqt. (7) asks is how does portfolio risk

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[2] $r=(P_1+D)/P_0$.

[3] In Sharpe's original paper, he shows the global investment frontier becoming flat along the capital market line so that competitive pricing of assets actually creates many "market" portfolios with various risk and returns that are linear combinations.

[4] Where all assets including i are held in value weighted proportions. These are the optimal ω_j's.

change as the efficient portfolio weight of asset *i* changes moving away from the efficient weight or from an inefficient combination toward the efficient one.

Mathematically, eqt. (7) evaluated at $\omega = 0$ reduces to only a few terms:

$$\left.\frac{\partial \sigma_p}{\partial \omega}\right|_{\omega=0} = \frac{[\sigma_{im} - \sigma_m^2]}{\sigma_m} \qquad 8$$

The result is that the change in portfolio risk in the neighborhood of the efficient frontier is equal to the difference between the covariance of asset *i* with the market minus the variance of the market divided by the standard deviation of the market. The slope of the bullet that makes up the efficient frontier at the market portfolio can be written as the ratio of eqt's (6) and (8):

$$\left.\frac{\partial E(r_p)}{\partial \sigma_p}\right|_{\omega=0} = \frac{\dfrac{\partial E(r_p)}{\partial \omega}}{\left.\dfrac{\partial \sigma_p}{\partial \omega}\right|_{\omega=0}} = \frac{E(r_i) - E(r_m)}{\dfrac{[\sigma_{im} - \sigma_m^2]}{\sigma_m}} \qquad 9$$

Equation (9) tells us the slope of the bullet as the portfolio share of asset *i* approaches efficiency. From the capital market line, we know the slope of the efficient frontier. From (1) we can write:

$$\frac{E(r_m) - r_f}{\sigma_m} \qquad 10$$

As the slope of the bullet show by eqt. (9) approaches efficiency, it approaches the slope of the capital market line. At the efficient share for asset *i*, equations (9) and (10) are equal. By linking eqt's (9) and (10), we can deduce what happens as the portfolio share of asset *i* converges to the optimal.

$$\frac{E(r_i) - E(r_m)}{\dfrac{[\sigma_{im} - \sigma_m^2]}{\sigma_m}} = \frac{E(r_m) - r_f}{\sigma_m}$$

Rewriting gives:

$$[E(r_i) - r_f] = \frac{\sigma_{im}}{\sigma_m^2}[E(r_m) - r_f] = \beta_i [E(r_m) - r_f] \qquad 11$$

Equation (11) is it—the Capital Asset Pricing Model, or CAPM for short. It tells us that the expected return on asset *i* net of the risk free return is *linearly* related to the expected market return net of the risk free. The implication is that if the market return increases, then the return to asset *i* must increase as well, and the amount that it will increase is equal to the increase in the market return times the linear correlation of the market return and the *i*th asset's return.

Equation (11) is represented by the familiar **market model**, which is usually seen in the common form

$$r_i = \alpha + \beta r_m + \varepsilon_i, \qquad 12$$

where $\alpha = (1-\beta)r_f$, and β is found by the least-squares estimator, $\hat{\beta} = \frac{\hat{\sigma}_{im}}{\hat{\sigma}_m^2}$.

Linear Pricing and Convergence to an Equilibrium

It is enlightening to reflect on the nuances of the CAPM that are revealed when we consider what the model says if there is no risk free asset. With no risk free, investors face the global investment frontier. Based on their own preference functions they attempt to form portfolios that are idiosyncratic in their composition. That is, a relative risk taker will choose a portfolio composed from more high return and high risk assets, while a relative risk averter will choose more low return low risk assets. When there is a risk free asset, all investors choose the same portfolio of risky assets.

With no risk-free asset, when investors are choosing among alternative portfolios of risky assets, the amount that they are willing to pay for assets differs. Relatively risk averse consumers will be willing to pay slightly more for low risk assets, while relatively risk loving consumer will be willing to pay slightly more for high risk assets. As relative risk lovers bid down the return on high risk stocks, it changes the investment frontier that faces relative risk averters. Similarly, as relative risk averters readjust their portfolios, it changes the frontier yet again.

The problem is that there is no assurance that with different investors searching for different portfolios along the global investment frontier, assets will be priced consistently. Assets are still valued based on their relative covariances. However, the relative rankings may not be the same across all investors. The linear pricing between any one asset and all the rest disappears.

Some Notes on Compound Returns

Wealth accumulation over the holding period can be described in terms of the geometric compound formula:

$$\frac{P_n}{P_0} = \prod_{t=1}^{n} (1+r_t) = (1+r_H)^n$$

where r_t is the stochastic return on an asset during each segment of time, 1 to n, and r_H is a constant rate of interest that reflects the average of the stochastic return over the entire holding period, n. Rewriting gives us the holding period return explicitly:

$$r_H = \left[\prod_{t=1}^{n} (1+r_t)\right]^{1/n} - 1$$

which can also be expressed in terms of logs:

$$r_H = e^{\sum_{t=1}^{n} \ln(1+r_t)/n} - 1 = e^{\mu} - 1$$

Rewriting, taking logs, and expected values, we have:

$$E[\ln(r_H + 1)] = E[\sum \ln(1+r_t)/n] = \mu$$

which says that the expected value of the log of the holding period return plus one is equal to the expected value of the average of the log of the stochastic return and this is equal to a constant, mu.

Let $(1+r_t) = x$, which is a random variable such that its natural log, ln(x), is distributed normally with mean, μ, and standard deviation, σ. The density of x is then called the log-normal. It looks similar to the normal with some adjustments:[5]

$$f(x) = \frac{1}{x\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}[\ln(x)-\mu]}$$

The median of $(1+r_t)$ is e^{μ} and the expected value[6] of $(1+r_t)$ is $E(x) = e^{\mu+\sigma^2/2}$. From the expected value, we can write:

$$\mu = \ln(E(x)) - \frac{1}{2}\sigma^2$$

which says:

$$E\ln(1 + r_t) = \ln(1+E(r_t)) - \sigma^2/2$$

From this we can calculate the holding period return from the average of the stochastic returns:

$$r_H = e^{[\ln(1+E(r))-\sigma^2/2]} - 1$$

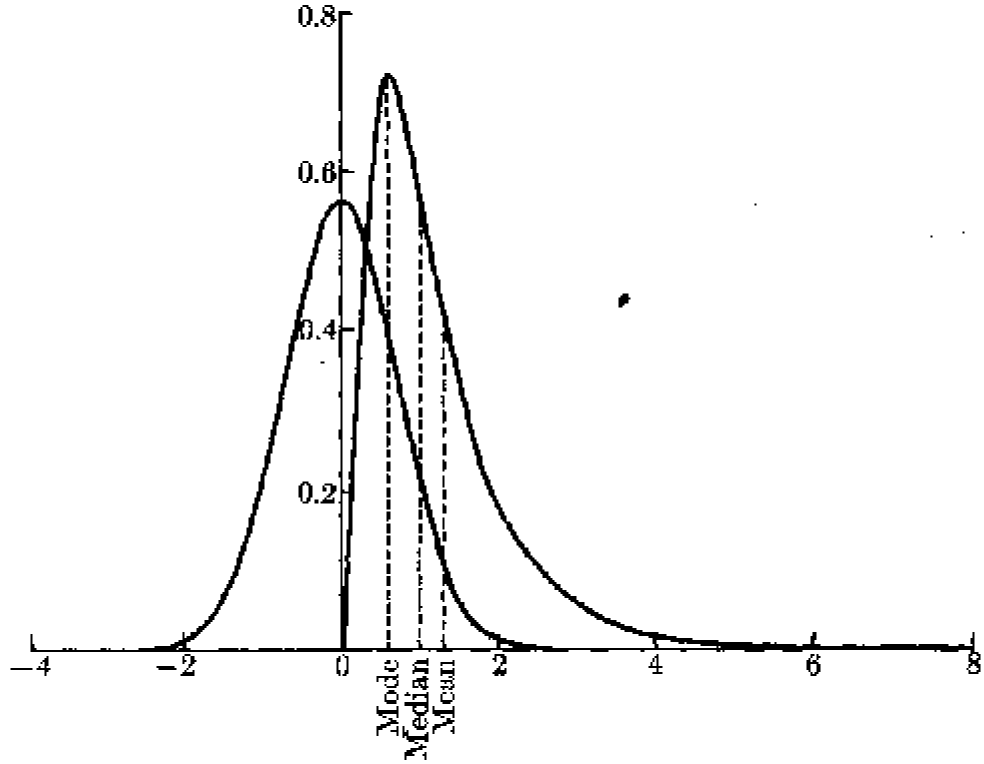
This makes intuitive sense because the anti-log function compresses the negative tail of the density and skews the distribution. The average of the stochastic returns is higher than the holding period returns because of the skewness and this is accounted for by subtracting one-half the variance.

Since σ is the standard deviation of log($1+r$) we need a transformation there also. This gets a little messy, but for small value of σ, the standard deviation of x and of ln(x) are very close. By small values, I mean .1 or less. If the standard deviation of r is .1, then σ is .091; if the standard deviation of r is .05, then σ is .0497. This means that you can just use the standard deviation of r in the calculation. The exact formula is:

$$V(r) = e^{2\mu+\sigma^2}(e^{\sigma^2} - 1)$$

Figure 1: Normal and Log-Normal Density

[5] If $x \sim f(x)$ then $g(x) \sim g'(x) f(g(x))$, where $g()$ is a monotonic transformation.
[6] The expected value can be derived from a moment generating function.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Mode
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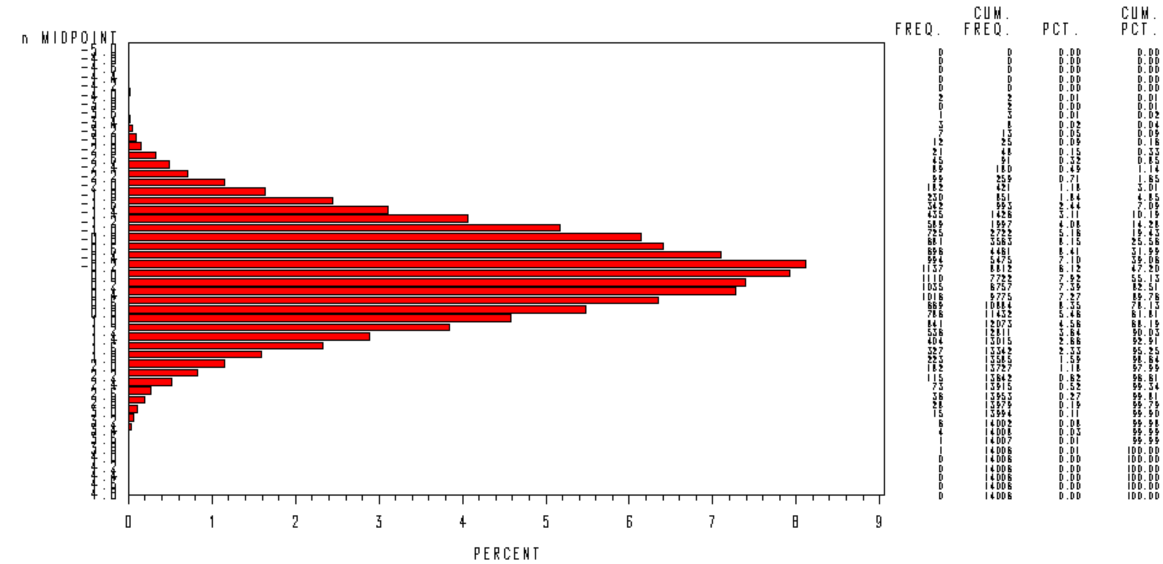
Some numbers on S&P 500 Index including dividends, 1/1/1950 -- present.

			Annualized
Average daily returns:		.0358%	8.767%
Std. dev.:		.00896	
Compound return:		.0307%	8.040%
Estimated compound return:		.0308%	
Number of days [1/3/'50-9/7/'05]:	14009		
Average monthly return on S&P 500:		.728%	8.733%
Standard deviation of monthly returns:		.0399	
Compound return:		.648%	8.059%
Estimated compound return:		.648%	
Number of months [1/3/'50-8/31/'05]:	667		
Average annual return on S&P 500:		9.387%	
Standard deviation of annual returns:		.1676	
Compound return:		8.081%	
Estimated compound return:		7.861%	
Number of years ['50-'04]	54		

The annualized returns are derived using the compound formula from the compound return from the shorter period, i.e., the annualized value for monthly returns is (1+monthly compound return) to the power of 12.

See below the graph of the unit normal variate of daily returns compared to the true unit normal distribution. The picture truncates the largest values of the returns and still the pictures differ markedly.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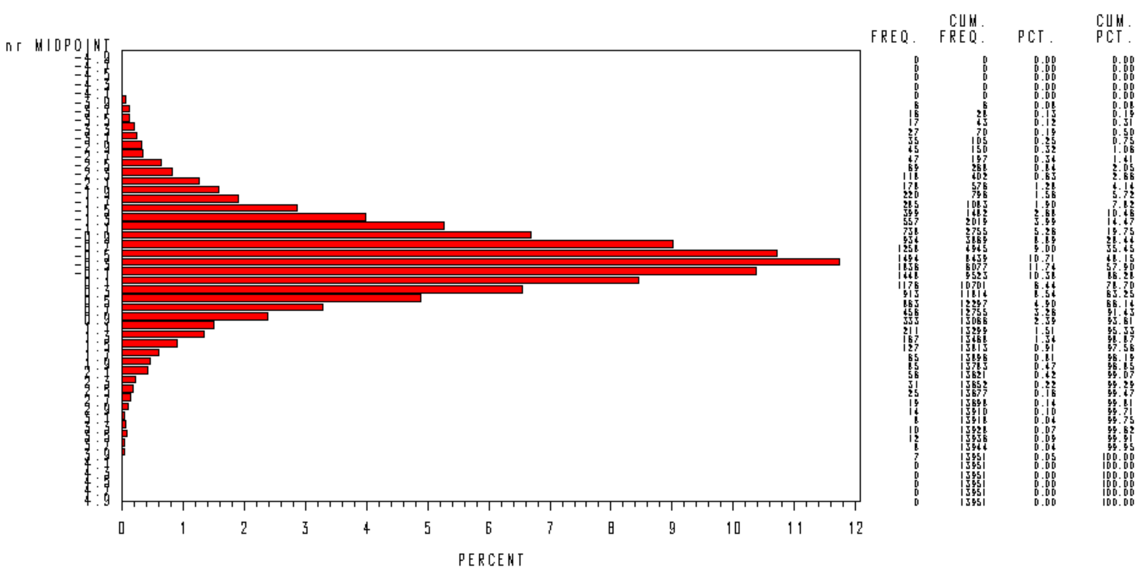
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