Summary & Review of the Capital Asset Pricing Model

The fundamentals of the derivation of the CAPM can be outlined as follows:

(1) Risky investment opportunities create a "Bullet" of portfolio alternatives. That is, each investment opportunity stands by itself. It has an expected return and some risk attached to it. However, if the investor/consumer chooses to invest in multiple projects, in their combined form, the portfolio return and risk are different than any one in isolation. Indeed, portfolio investments generate lower risk for any level of expected return than can be achieved by investing in a single project. This is the process of *diversification*.

(2) The Bullet of risky portfolio choices can be linked to the risk-free return to create the Capital Market Line. The CML is the efficient frontier of investment opportunities because it employs not only the diversification benefits of portfolio investing but also the risk hedging potential of the risk-free return to create the highest possible expected return for any given level of risk.

An excellent illustration of the power of diversification is found by drawing a bullet of two assets such that the lower return asset actually has a return that is less than the risk free return. The obvious question is posed: Why would an investor hold an asset which is risky which has a return that is less than the risk free. The answer is because the risky asset has portfolio diversification power. Holding it reduces the portfolio risk associated with other risky assets in a way that cannot be achieved with the risk-free asset. Portfolio diversification is truly as close as we come in the social sciences to magic.

(3) Given the Bullet and the CML, all assets are priced by the equality of the slope of the Bullet and the CML. That is, all assets have to offer an expected return given their risk that make them attractive to investors who are choosing among other risky assets and the risk-free return.

To see the pricing phenomenon of the CAPM, let's consider a comparative static experiment.[1] Let's consider the effect on the pricing of assets that occurs when the risk-free return changes. Assume that there are only two risky assets, X and Y. Start with a value for the risk-free return of r_f^0. There is an efficient portfolio of {X,Y} designated as point *a*. That is, point *a* tells us the optimal portfolio allocation of invested wealth between X and Y. **{Graph to come.}** The consumer/investor then chooses how much risk to bear by holding more or less wealth in the risk-free asset.

Next, let the risk-free asset increase from r_f^0 to r_f^1. A line drawn from the new risk-free rate to the old bullet of X and Y is tangent at point *b*. Point *b* is not an equilibrium, but rather depicts a "tendency." Given the new risk-free rate, investor/consumers are now interested in holding more of asset Y and less of asset X. *This is the prediction of the CAPM.*

Because of the change in the risk-free rate, investors will optimally sell asset X and buy asset Y. Selling X causes its price to fall. Buying Y causes its price to rise. When the price of X falls, its expected return increases. When the price of Y rises, its expected return decreases. In risk-return space, point X shifts up and point Y shifts down.

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[1] That is, let's consider an experiment using CAPM akin to the experiments that we employ in Supply and Demand analysis. In S&D analysis we shift the demand curve and then describe how the equilibrium price changes.

The Bullet rotates. There is a new equilibrium tangency between the CML and the Bullet. Label this point *c*. Point *c* is the efficient allocation of investment wealth between assets X and Y, given the risk-free rate r_f^1. At point *c*, investors hold more Y and less X than they did at point *a*. But just like point *a*, at point *c*, assets X and Y are priced based on the equality of the slope of the Bullet and the slope of the CAPM.

The CAPM explains how buying and selling pressure on assets will reach an equilibrium. (This is just the same way that S&D analysis tells us how buying and selling pressure in commodity markets reaches an equilibrium.) The CAPM tells us when utility maximizing investor/consumers will buy and sell assets. It tells us how that buying and selling activity will diminish as an equilibrium is approached. And, it gives us an equilibrium condition, i.e., it describes a situation in which there is no incentive to buy or sell assets.

The price level of assets is based on the expected future cash flows that those assets will enjoy relative to the cash flows of other assets. The CAPM assumes that the joint distribution of cash flows is independent of the current price of the assets. Hence, the current prices adjust relative to one another until the CAPM equilibrium is reached. This is shown in the picture by the vertical shifts in X and Y as the risk-free return changes. The exogenous change in the risk free causes consumers to reallocate their portfolio of risky assets. This portfolio adjustment changes the level of the asset prices, but it does not change the distribution of the future cash flows. Hence, an equilibrium is reached between (among in the *n* asset case) prices based on their expected returns and the unchanging distribution of cash flows. The distribution of cash flows is defined by the standard deviation of cash flows to the two assets separately, and the covariance between the cash flows of each pair.

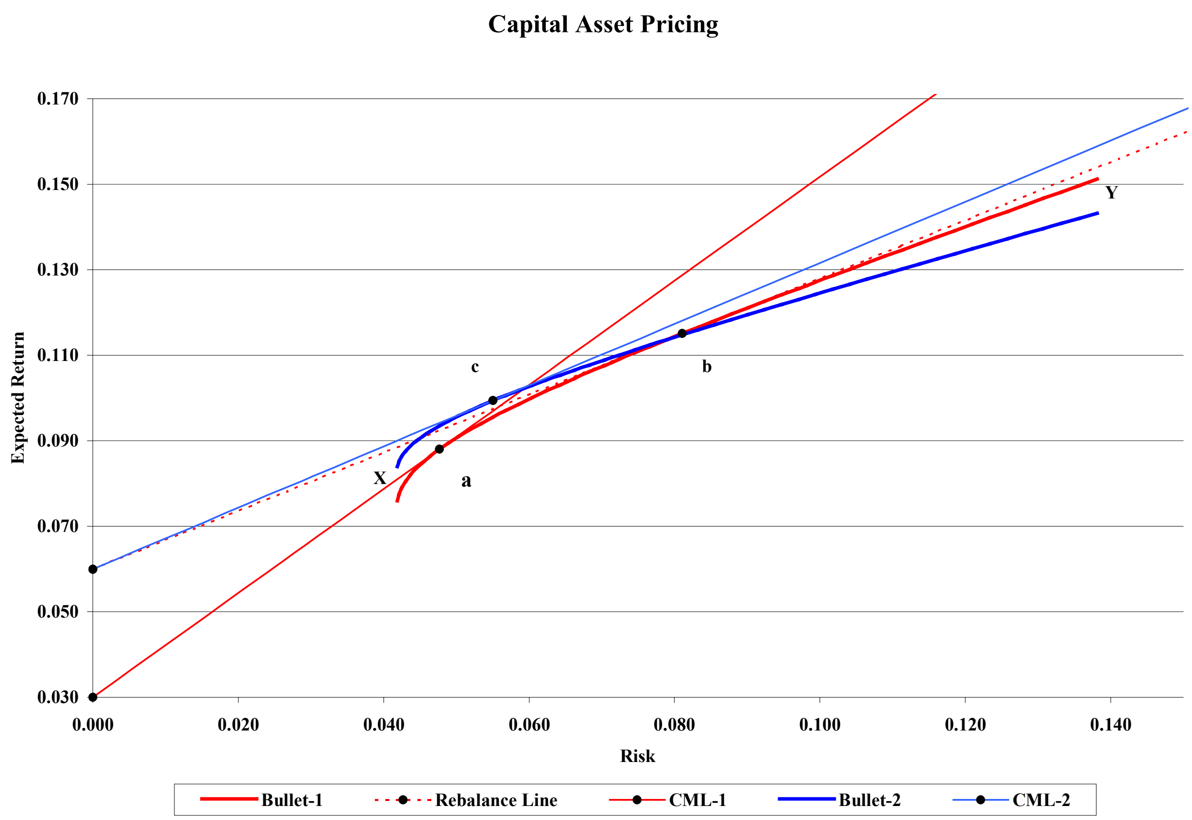
The CAPM prices assets looking forward. That is, the CAPM price of assets is based on the expectation of future cash flows. We commonly use the past as a prediction of the future, but it is not necessarily a perfect predictor. In this sense the standard expression of the CAPM in the market model, $r_i = \alpha_i + \beta_i r_m + \varepsilon_i$, is based on the idea that β is deterministically identified by the joint distribution of future cash flows. Our estimation of β using past observations on r_i and r_m is just an estimate which has its own sampling errors and potential errors of measurement.

A common question that bright students ask is, How can one make money using the CAPM? The proper answer is the following: Assuming that you have an accurate measure of β, if you find assets priced such that

$$[E(r_i) - r_f] > \beta_i[E(r_m) - r_f]$$

buy. If the asset is priced such that the inequality goes the other way, sell. If you can find assets exhibiting these characteristics, you can make a profit. The problem is that you will not likely find many because that is what all investors are trying to do.

One final point, and a point to which we will return, the E(r_i) given by the CAPM is the interest rate by which the cash flows of firm *i* are efficiently discounted in the DCF formula.


Capital Asset Pricing
Expected Return
0.170
0.150
0.130
0.110
0.090
0.070
0.050
0.030
0.000
0.020
0.040
0.060
0.080
0.100
0.120
0.140
Risk
X
a
c
b
Y
Bullet-1
Rebalance Line
CML-1
Bullet-2
CML-2

Empirical Tests of the CAPM:

The model is constructed as:

$$(r_{it} - r_f) = \gamma_0 + \gamma_1 (r_{mt} - r_f) + \gamma_2 (r_{mt} - r_f)^2$$

where r_f is either the return on Treasury bills or the return on a zero beta portfolio. The model is estimated over various periods, and different specifications for *i*. Most researchers treat *i* as a portfolio of assets. The CAPM prediction is that γ_0 and γ_2 should be zero, and γ_1 is the value of β for asset group *i*.

Generally, researchers find that γ_2 is zero but that γ_0 is not.

Example of the analysis using Rodgers' data: {Dell, Google, Nike, Pfizer, Sony}, the S&P 500, and the 20 yr constant maturity gov't bond yield.

```
data one; set one;
rf=(1+ir/100)**(1/365)-1;
netr=sp-rf; netp=portfolio-rf;
netr2=netr**2;
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                                    The SAS System                                   10
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                                  The MEANS Procedure
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Variable	Label	N	Mean	Std Dev	Minimum	Maximum
Date	Date	256	16505.58	107.2588271	16322.00	16691.00
Portfolio	Portfolio	256	0.000900053	0.0087706	-0.0214747	0.0269997
SP	SP	256	0.000415645	0.0064548	-0.0167205	0.0197363
ir	VALUE	253	4.6910277	0.2039874	4.2500000	5.0700000
netp		253	0.000662007	0.0087009	-0.0216013	0.0268731
netr		253	0.000254051	0.0064691	-0.0168471	0.0196096
netr2		253	0.000041749	0.000056072	2.4189159E-9	0.000384538
rf		253	0.000125601	5.3406143E-6	0.000114038	0.000135507

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                          The SAS System                                    9
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                         The REG Procedure
                           Model: MODEL1
                     Dependent Variable: netp

          Number of Observations Read                        256
          Number of Observations Used                        253
          Number of Observations with Missing Values           3


                        Analysis of Variance

                                 Sum of           Mean
Source                   DF     Squares         Square    F Value    Pr > F

Model                     2     0.00761        0.00381      82.97    <.0001
Error                   250     0.01147     0.00004587
Corrected Total         252     0.01908


           Root MSE              0.00677    R-Square     0.3989
           Dependent Mean     0.00066201    Adj R-Sq     0.3941
           Coeff Var          1023.03763


                        Parameter Estimates

                        Parameter       Standard
     Variable     DF     Estimate          Error    t Value    Pr > |t|

     Intercept     1   0.00070672     0.00053124       1.33      0.1846
     netr          1      0.84995        0.06600      12.88      <.0001
     netr2         1     -6.24315        7.61462      -0.82      0.4131
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Fama and French multi-factor models:

$$E(r_P) - r_f = b[E(r_m) - r_f] + sE(SmB) + hE(HmL)$$

CAPM is augmented by *SmB*, which is the difference in the returns to small versus large capitalized stocks, and *HmL*, which is the difference in the returns to high versus low book-to-market stocks.

These are so-called mimicking portfolio returns built on firm size and the ratio of book to market equity, respectively. A mimicking portfolio return is a return that is the difference between the returns to two classes of firms. For instance, in the size mimicking portfolio return, the returns of big firms are subtracted from the returns to small firms. The result is a portfolio return differential that reflects the commonality of returns to big firms as that is different from returns to small firms, and both net of the return to all firms, which is held constant by the relation between the i^{th} firm's return and the market.

The coefficients on the mimicking portfolios describe common factors in returns across firms if the coefficients consistently associate firms with their peers. That is, a given big firm should have negative return when *SmB* is positive; likewise, when *SmB* is positive, a small firm should have a positive return. The implication is that s should be positive for small firms and negative for big firms if size is a factor common in the rational risk pricing of assets.

Similarly, *HmL* is a portfolio return differential between high book-to-market firms and low book-to-market firms. If book-to-market is a common risk factor, h should be positive for high book-to-market firms and negative for low book-to-market firms.

On the other hand, if CAPM captured all attributes of asset pricing, then s and h should be zero and b would be the portfolio beta.

Fama and French show that size and book-to-market are indeed common risk factors.

Roll's critique: There is no market portfolio; hence, there can be no test of CAPM. Roll's critique is interesting because if there were a true market portfolio, there would be no need to rebalance portfolios. Everyone would hold everything and as everything was repriced individuals portfolios would automatically rebalance. Because there is no market portfolio, it is necessary to rebalance as expected cash flows and relative risk change. And we know there is a lot of rebalancing.

In some ways the question about the accuracy of CAPM depends on the question. I argue that CAPM is a predictor of an equilibrium repricing process. While not necessarily perfectly accurate, it indicates the direction that prices will move. Hence, here are a couple of "tests" of CAPM that I think are appropriate:

1) Estimate beta up to day t-1. Use beta to predict the direction of change in each stock based on the direction of change of the average of all stocks. Also estimate the magnitude. High beta stocks should change more in absolute value than low beta stocks.

2) Compare changes in stock prices to changes in the risk free rate. As the risk free increases, low beta stock prices should fall and high beta stock prices should rise.

Revisiting CAPM — Chris Kirby

The purpose of this lecture is to revisit the capital asset pricing model by more explicitly examining (1) Expected Utility Maximizing and (2) Portfolio Optimization. As we have shown before, we will demonstrate that utility maximizing consumers will choose to hold a single portfolio of risky assets, which then will result in equilibrium pricing of all assets.

Let's start with a specific utility function, i.e., the negative exponential:

$$U(W) = -e^{-aW}$$

This function has several properties that are useful. Most importantly, it is convenient. Moreover, it has constant absolute risk aversion, which is identified in the parameter, *a*.

$$ARA = -U'' / U' = a^2 e^{-aW} / ae^{-aW} = a$$

The shape of the utility function is upward sloping and concave. It starts in the negative quadrant at -1 and asymptotes to zero.

Portfolio Optimization:

Consider the consumer/investor's choice between two assets. One is risky; the other is riskfree. The risky asset has return R with a expected value of μ and a standard deviation of σ. The riskfree has a known return of r_f.

Let x be the proportion of the portfolio wealth that is held in the risky asset. For simplicity, allow the initial value of the portfolio to be 1. The portfolio return can then be written as:

$$R_P = xR + (1-x)r_f$$

or

$$R_P = r_f + x(R - r_f)$$

where the expression in parentheses is called *excess return*. The expected return on the portfolio is:

$$E(R_P) = E(r_f + x[R - r_f]) = r_f + x(\mu - r_f)$$

and the variance of the portfolio return is:

$$Var(R_P) = x^2\sigma^2$$

Maximizing Expected Utility:

The consumer maximizes expected utility over the choice of x:

$$\max_{\{x\}} E(U(W)) = E\left(-\exp(-a[r_f + x(R - r_f)])\right)$$

To operationalize this expression, we assert that the return on the risky asset follows some distribution. Thus, we can write:

1. $$\max_{\{x\}} E(U(W)) = \int -\exp(-a[r_f + x(R - r_f)]) \cdot f(R; \mu, \sigma^2, \theta) dR$$

for some general density of R.[2] It is extremely useful to assume that the density $f(.)$ is the normal. This is true because of two things: (a) The normal is fully described by its mean and standard deviation, so the extra parameter θ vanishes. We will see later that this becomes a crucial assumption. (b) The other reason that it is useful to assume that the return on the risky asset is distributed normal is because it allows for a very simple expression of expected utility.

For any random variable that is normally distributed, i.e., y~$N(m,s^2)$, we know that:

$$E(e^{cy}) = e^{cm + \frac{c^2 s^2}{2}}$$

Thus, the expected utility of wealth given by (1) above can be written as:

2. $$E(U(W(x))) = -\exp\left(-a[r_f + x(\mu - r_f)] + \frac{a^2 x^2 \sigma^2}{2}\right)$$

The consumer/investor maximizes (2) by the choice of x. The FOC looks like:

$$\frac{\partial E(U(W(x)))}{\partial x} = [a(\mu - r_f) - x a^2 \sigma^2] \exp(.) = 0$$

which gives the optimal portfolio choice:

3. $$x^* = \frac{1}{a} \frac{(\mu - r_f)}{\sigma^2}$$

Equation (3) makes sense. The consumer adjusts her portfolio toward more risky assets as her degree of risk aversion, a, declines. Riskiness is defined in terms of the excess return divided by the variance of the return on the risky asset. All consumer/investors are characterized by (3).

[2] In general, the density will be described by its moments where we use θ to stand for the higher moments.

Notice that if we graph portfolio choice in expected return and risk space, $\{E(R)$, Std Dev.$\}$, we have simply a straight line starting at r_f on the vertical axis and going through the point $\{\mu,\sigma\}$. The consumer's choice of x determines where she is on this line. For consumers that are very risk averse (high a) lie somewhere to the left of $\{\mu,\sigma\}$. Consumers who are not very risk averse can borrow at the riskfree rate so that they are to the right of $\{\mu,\sigma\}$.

Notice that they way that we have developed this analysis, $\{\mu,\sigma\}$ is just a point in risk-return space. Now consider what happens if the consumer has choices among risky assets. That is, assume that there are many $\{\mu,\sigma\}_i$; these are described by the many risky assets in the world and by the investment frontier of possible portfolio choices, that is, the *bullet*. Which risky portfolio will the consumer choose.

This is the same thing as asking which $\{\mu,\sigma\}$ maximizes:

$$E(U(W(x^*))) = -\exp\left(-a[r_f + x^*(\mu - r_f)] + \frac{a^2 x^{*2} \sigma^2}{2} \right)$$

Clearly the expected utility of wealth is maximized by choosing the *tangent portfolio*, that is the portfolio with the highest excess return relative to its risk: $\frac{\mu - r_f}{\sigma}$.

So we are back to the plank that connects the riskfree asset to the risky investment frontier. All consumer/investors will choose the same risky asset in order to maximize their expected utility of wealth. This risky asset will be a portfolio of all risky assets. All consumer/investors hold the same risky portfolio and vary the proportion of their wealth held in this portfolio and the proportion held in riskfree asset based on their degree of risk aversion.

Importantly, because all individuals hold the same portfolio of risky assets, all assets are valued the same by all individuals. Thus, we have a pricing equilibrium: CAPM.

Recognize the degree to which this equilibrium theorem is built on the assumption of the normal distribution. By assuming normality in the return to the risky asset we are able to describe the distribution of returns and the expected utility of wealth in terms of two parameters, $\{\mu,\sigma\}$, and all individuals have the same valuation of these parameters. If we construct the theory based on a distribution other than the normal, the higher moments of distribution will enter the utility function and these will be valued differently based on each individual's degree of risk aversion. Hence, we will not have an equilibrium pricing model.