The Application of Asset Pricing to Portfolio Management

The Nature of the Problem

Portfolio managers have two basic problems. First they must determine which assets to hold in a portfolio, and second, they must determine how much of each asset to hold as a fraction of the total portfolio value. Both questions have time dimensions: Both questions must be answered when the portfolio is initially formed. Then, at each moment thereafter, the portfolio manager must make a decision about whether to rebalance the portfolio or simply hold the assets initially purchased.

The purpose of this discussion is to compare different portfolio construction and adjustment strategies theoretically and empirically. We address the following questions:

- For a portfolio of *n* risky assets, what initial portfolio allocations are best?
- As the values of the *n* assets change, what efficiencies result from rebalancing the portfolio?
- For a portfolio composed of *n* risky assets, what is the value of one additional asset?
- What rules should be used in choosing the *n* risky assets?

Alternative Portfolio Construction

Consider the problem of managing a small portfolio of assets in which the investor wishes to achieve a target rate of return with minimum variance. There are several approaches to this problem that one might adopt. A common textbook assumption is that the portfolio manager puts equal dollar amounts of the portfolio capital into *n* risky assets. The number and type of assets in the portfolio and the relative proportion of risky assets to the risk free investment determines the expected return and the level of risk when the portfolio is initially formed. This assumption is used in textbooks to lay the theoretical foundation for the relation between risk and the number of assets held in a portfolio. While simple as a theoretical device, this equal-weight strategy becomes complicated as a practical portfolio-management rule. As the value of each of the *n* risky assets changes, the portfolio is no longer composed of *n* equal weights. The question, then, is whether and how often the portfolio should be rebalanced.

A slightly more elaborate portfolio allocation rule is one that determines portfolio weights based on the capitalized value of the assets held in the portfolio. This is called a value-weighted portfolio. That is, the portfolio manager chooses *n* assets in which to invest. The total portfolio capital is then spread across the assets based on the total value of each asset relative the total value of all *n* assets. Say an investor wants hold $4000 worth of wealth in the stock of two telephone companies, BellSouth and Nynex. In terms of equity, BellSouth is a $30 billion company. Nynex has an equity value of $10 billion. To hold a value-weighted portfolio, $3000 is held in BellSouth stock and $1000 in Nynex securities.

The value weighted portfolio approach is a buy-and-hold strategy. Since the total portfolio wealth is divided across the assets in proportions based on the total capitalized value of each asset, as the values of the assets change, so too does the share of the portfolio devoted to each. Using this weighting scheme, the portfolio is always balanced as the values of the assets change. No portfolio adjustment is required.

This simple buy-and-hold, value weighting strategy has been the benchmark against which mutual fund performance has typically been judged: The consensus of the academic literature seems to be that mutual fund rebalancing adds little to performance relative to the value-weighted/buy-and-hold benchmark.[1] Nonetheless, portfolio managers do a substantial amount of rebalancing, and, in a competitive environment, the *a priori* expectation must be that survival implies efficiency. One of the goals of this exercise is to better understand this anomaly.

An alternative to equal- or value-weighting is to choose portfolio weights based on parameters derived from asset pricing models. The fundamental basis of asset pricing found in the theoretical literature is the Capital Asset Pricing Model (CAPM). In its simplest formulation, the CAPM says that assets will be priced by the market based on their covariance with other assets. Because of this, an individual investor can minimize risk for any level of expected return by dividing his wealth between a given amount of the risk free asset and a portfolio of *all risky assets*. When the individual investor holds a share in all risky assets, the weights on the assets are based on each asset's capitalized value. In this case, the theoretical implication of CAPM for portfolio management is the value-weighted/buy-and-hold approach.

However, when the number of risky assets held in a portfolio is a subset of all risky assets, the simple value-weighted/buy-and-hold method of portfolio management is not necessarily efficient. For portfolios containing a limited number of assets, simple CAPM as well as more sophisticated pricing models can be used to devise more efficient strategies for portfolio management. The next section of this paper shows that the parameters associated with these pricing models identify portfolio weights that are different from equal- or value-weighted assignments.

Using CAPM to Define Optimal Portfolio Weights

Assume that the investor wishes to hold a portfolio of risky assets that is a subset of all possible investments. The investor's objective is to minimize risk subject to a targeted, expected rate of return. CAPM is one way to identify both risk and the expected return on a risky asset. That is, for any risky asset

$$r_i = \alpha_i + \beta_i r_m + \varepsilon_i \tag{1}$$

where r_i is the return on asset i, r_m is the value weighted return on all risky assets, β is based on the covariance between the return on all risky assets and the return to asset i, α is equal to (1- β) times the return on the risk-free asset, r_f, and ε is the residual variation in the return to asset i. In the simple CAPM the systematic relation between assets is captured by the linear multiplier on the single market index. In more advanced models other indexes are used to identify the expected return on an asset. We will extend the model along those lines in a moment. First, we develop the optimal investment strategy using the simple CAPM.

[1] There is a recent paper by David Boothe and Eugene Fama in the *Financial Management Review* that does remark on the shortcomings of the buy and hold strategy.

The objective function of the investor can be defined as the desire to minimize portfolio variance for a given target portfolio return. This can be written as

$$\begin{aligned} &\min_{\{\omega_i\}} V(r_p) = \sum_{i=1}^{n}\sum_{i=j}^{n} \omega_i \omega_j \sigma_{ij} \\ &\text{subject to}: k = \sum_{i=1}^{n} \omega_i E(r_i) + \left(1 - \sum_{i=1}^{n} \omega_i\right) r_f \end{aligned} \quad \textbf{2}$$

where $V(r_p)$ is the variance of the portfolio return, the ω terms are the proportions of the portfolio devoted to each asset, and the σ terms refer to the variance and covariances of the returns on each risky asset. The risk free asset, which takes a portfolio share of one minus the sum of the shares devoted to the risky assets, is assumed to have no variance. The objective function is optimized subject to a target portfolio return, *k*.

The portfolio return is identified by using the CAPM; the expectation of ε is zero. The intercept of equation (1), α, is replaced by its theoretical identity involving β and the risk-free return, that is, $\alpha = (1- \beta)\, r_f$. For the constraint on the expected return, these substitutions give:

$$k = \sum_{i=1}^{n} \omega_i \left[(1 - \beta_i) r_f + \beta_i r_m \right] + \left(1 - \sum_{i=1}^{n} \omega_i\right) r_f \quad \textbf{3}$$

By canceling the terms involving the sum of the portfolio weights times the risk-free return, this can be simplified to:

$$k = \sum_{i=1}^{n} \omega_i \; \beta_i (r_m - r_f) + r_f\,. \quad \textbf{4}$$

Rewriting gives:

$$\sum_{i=1}^{n} \omega_i \beta_i = \frac{k - r_f}{r_m - r_f} = \phi \quad \textbf{5}$$

where the ratio of the target return net of the risk-free divided by the market return net of the risk-free is labeled ϕ.

The objective function can be rewritten by incorporating the CAPM terms so that it reduces to:

$$\min_{\{\omega_i\}} V(r_p) = \sum_{i=1}^{n} \omega_i^2 \sigma_{\varepsilon_i}^2 + \sigma_m^2 \left(\sum_{i=1}^{n} \omega_i \beta_i \right)^2 \quad \textbf{6}$$

where σ_m^2 is the variance of the market return. [See appendix.] This is a standard expression found in many books on portfolio theory.[2] The portfolio variance simplifies based on the following assumptions related to the CAPM. First, the covariance between the residual return (the ε_i's in equation 1) and the market return is zero, and second, the covariance of residual returns between individual assets is zero.[3] Importantly, the second assumption says that there is no expected covariance between any two risky assets except for the fact that they both will vary with the market return. This common response to overall market conditions is captured in β and there is no additional relation involved between the ε_i's.[4]

Equation (6) is optimized subject to the constraint given by equation (5). Using the method of Lagrange, we get the following first order conditions:

$$\frac{\partial V^L}{\partial \omega_i} = 2\,\omega_i\,\sigma_{\varepsilon_i}^2 + 2\,\sigma_m^2\left(\sum_{i=1}^{n}\omega_i\,\beta_i\right)\beta_i + \lambda\,\beta_i\,(r_m - r_f) = 0,\ \{i = 1,n\} \qquad \mathbf{7}$$

The set of equations defined by (7) can be solved in pairwise fashion to eliminate λ and the other constant terms. The ratio of the portfolio shares between any two assets readily reduces to:

$$\frac{\omega_i}{\omega_j} = \frac{\beta_i / \sigma_{\varepsilon_i}^2}{\beta_j / \sigma_{\varepsilon_j}^2} \qquad \mathbf{8}$$

The impact of the constraint can be included by multiplying by β_i and summing over n. This gives

$$\frac{\sum_{i=1}^{n}\beta_i\,\omega_i}{\omega_j} = \frac{\sum_i \beta_i^2 / \sigma_{\varepsilon_i}^2}{\beta_j / \sigma_{\varepsilon_j}^2}$$

From this, the value of the constraint given by equation (5) can be substituted directly to identify the value of each of weights:

---

[2] See for example, Wm. Sharpe, *Portfolio Theory*, 1970, McGraw-Hill.

[3] These assumptions are logically consistent with the CAPM so long as n is not the entire set of risky assets. If the portfolio included the set of all risky assets, then there would be an identity that operates across the set of functions shown by equation (1). That is, the sum of residuals across n-1 assets along with the market return would define the residual for the n^{th} asset. However, when n is a subset of all risky assets, this identity does not come into play.

[4] This assumption may seem extreme in the sense that, for instance, firms in the same industry may be expected to have returns that vary in relation to one another in a way that is more marked than described by the common variation with the market. However, the common variation between firms in the same industry can be either positive or negative. There will be events that cause all firms in an industry to become more valuable, but on the other hand, competition means that when one firm is better off, it reduces the profitability of its cohorts, at least marginally. A priori, we do not know which of these forces will dominate. In general, the whole idea of extending the CAPM is to identify and capture common variation in returns between firms, and one extension could well be an industry return factor. We will return to the theme of extending the pricing model shortly.

$$\omega_j = \frac{\phi}{\left[\sum_{i=1}^{n}\frac{\beta_i^2}{\sigma_{\varepsilon_i}^2}\right]}\frac{\beta_j}{\sigma_{\varepsilon_j}^2} \qquad \textbf{9}$$

It is important to recognize that the ω's are not necessarily positive. When they are negative, it implies that the investor takes a short position in an asset. This will be the case when one of the assets has a negative β. Nor do the weights sum to one. For instance, consider the case where ϕ is equal to one. In this case, the investor chooses a target return that is the market. However, given the set of *n* stocks held by the investor, the market return may only be available by borrowing at the risk free rate and investing this additional amount of capital in the portfolio of *n* risky assets. The general result is given by summing equation (9) over *n*:

$$\sum_{i=}^{n}\omega_i = \phi\frac{\sum_{i=1}^{n}\frac{\beta_i}{\sigma_{\varepsilon_i}^2}}{\sum_{i=1}^{n}\frac{\beta_i^2}{\sigma_{\varepsilon_i}^2}}$$

Extending the CAPM

The set of portfolio weights defined by equations (8) and (9) are based on the values of β and residual variance for each asset that are derived from the asset pricing model. The efficiency of the portfolio weights in this scheme can be improved by refining the pricing model. One such an extension of the pricing model is developed in recent research by Fama and French. In their model equation (1) becomes

$$r_i = \alpha_i + \beta_i r_m + \gamma_i M_s + \delta_i M_b + \varepsilon_i \qquad \textbf{10}$$

where M_s and M_b are so-called mimicking portfolio returns built on firm size and the ratio of book to market equity, respectively. A mimicking portfolio return is a return that is the difference between the returns to two classes of firms. For instance, in the size mimicking portfolio return, the returns of small firms are subtracted from the returns to big firms. The result is a portfolio return differential that reflects the commonality of returns to big firms as that is different from returns to small firms, and both net of the return to all firms, which is held constant by the relation between the *i* th firm's return and the market.

The coefficients on the mimicking portfolios describe common factors in returns across firms if the coefficients consistently associate firms with their peers. That is, a given big firm should have positive return when M_s is positive; likewise, when M_s is positive, a small firm should have a negative return. The implication is that γ should be positive for big firms and negative for small firms if size is a factor common in the rational risk pricing of assets. Similarly, M_b is a portfolio return differential between high book-to-market firms and low book-to-market firms. If book-to-market is a common risk factor, δ should be positive for high book firms and negative for low book firms. Fama and French show that size and book-to-market are indeed common risk factors.

For the purpose of optimal portfolio balancing, the extended CAPM given in equation (10) complicates matters somewhat. The variance of the portfolio return becomes

$$\min_{\{\omega_i\}} V(r_p) = \sum_{i=1}^{n} \omega_i^2 \sigma_{\varepsilon_i}^2 + \sigma_m^2 \left(\sum_{i=1}^{n} \omega_i \beta_i \right)^2 + \sigma_s^2 \left(\sum_{i=1}^{n} \omega_i \gamma_i \right)^2 + \sigma_b^2 \left(\sum_{i=1}^{n} \omega_i \delta_i \right)^2 \qquad \mathbf{11}$$

This formulation is based on several assumptions. Again, we treat the covariance between the residuals of the pricing equations across assets as zero and assume that the residuals are not correlated with the market and risk factor indexes (i.e., the market return and the mimicking portfolio returns on size and on book-to-market equity). Furthermore, we assume that the indexes are not themselves correlated. Thus, the portfolio variance for the *n* assets is given by equation (11).

Even though the portfolio variance becomes more complex when we expand the pricing model, the constraint is unchanged. Because the mimicking portfolio returns are differences between the returns to firms based on the characteristics of size and book-to-market, their expectation is zero. For instance, looking ahead we have no reason to expect that big firms will outperform small firms or vice versa. Therefore, when we compute the constraint on the expected portfolio return, the mimicking portfolio indexes fall out.

The first order conditions for the optimization problem using the extended pricing equations reduce to equation (12) in terms of the relation between any two portfolio weights:

$$\omega_i = \frac{\beta_i / \sigma_{\varepsilon_i}^2}{\beta_j / \sigma_{\varepsilon_j}^2} \omega_j + \frac{\sigma_s^2}{\sigma_{\varepsilon_i}^2} \left(\sum_{k=1}^{n} \omega_k \gamma_k \right) \left(\frac{\gamma_j}{\beta_j} - \frac{\gamma_i}{\beta_i} \right) + \frac{\sigma_b^2}{\sigma_{\varepsilon_i}^2} \left(\sum_{k=1}^{n} \omega_k \delta_k \right) \left(\frac{\delta_j}{\beta_j} - \frac{\delta_i}{\beta_i} \right), \ \forall i, j. \qquad \mathbf{12}$$

The constraint imposes the additional requirement on the overall size of the weights; it is the same as in the simple CAPM as given by equation (8). Notice that the solution value for equation (12) is complicated by the fact that the sum of the weights times the mimicking portfolio coefficients shows up for both indexes. Because of this, the formula for the portfolio weights given by (12) do not have the simple solution described by equations (8) and (9) where we used the basic CAPM. Nonetheless, a unique solution is attainable using numerical methods.

Preliminary Empirical Results

Equations (8), (9), and (12) define rules for constructing portfolios. The rules apply where portfolios contain a limited number of assets. The question is whether these rules actually improve portfolio performance. The benchmark is to compare the performance of portfolios constructed using these rules to the performance of value-weighted portfolios.

In order to asses the value of the portfolio construction strategies described above, we simulated returns to 100-stock portfolios using NASDAQ securities. We used returns to securities traded in the NASDAQ National Market System (NMS) over the years 1984 through 1991.

Using these returns we estimated for each stock the simple CAPM as well as the extended pricing models like those described by Fama and French for each of the years 1984 through 1990. (The estimation procedure and diagnostic analysis is described elsewhere.) Given the pricing model estimates, we formed portfolios for the following year. The returns and variances of these portfolios were then compared to the returns and variances of valued-weighted portfolios of the same securities.

The preliminary findings of this research suggests that the portfolio balancing approach based on pricing model parameters does, in fact, produce superior performance compared to a value-weighted/buy-and-hold strategy. Further work is needed to assess the magnitude and sensitivity of this result.

More Evidence on the Effect of Portfolio Diversification

The following is a table concerning the effects of portfolio diversification:

Table 1: Optimal v. Sub-optimal Portfolios

Market Variance	0.001043	0.001043	0.001043	0.001043
Number of Assets	15	25	45	85
Variance of Optimized Portfolio	0.001581	0.001479	0.001260	0.001167
Inefficiency of Optimized Portfolio	0.000538	0.000436	0.000217	0.000125
Inefficiency %	51.56%	41.82%	20.85%	11.95%
Sub-optimal Portfolio Variance	0.002329	0.001954	0.001526	0.001284
Inefficiency of Sub-optimal Portfolio	0.001286	0.000911	0.000483	0.000241
Inefficiency %	123.28%	87.34%	46.29%	23.15%
Reduction in Inefficiency	58.18%	52.12%	54.97%	48.40%

% of Optimal Portfolio held in Risky Assets	55.5%	61.7%	72.5%	84.2%

This table is constructed from the another mutual fund data set, which is available from my hard disk in the subdirectory C:\pub. The file is '855b1.wk1'. On this file, the betas and R^2's for each fund are given along with the ten year average return and standard deviation. These data come from Morningstar. I constructed the standard error of the residual of the market model and included it in the spreadsheet. The formula is simple:

$$\sigma_\varepsilon^2 = (1 - R^2)\sigma_r^2$$

Thus given the standard deviation of the fund and the goodness of fit of it to the market model, we can compute the residual variance.

Next I solved for the optimal portfolio weights using portfolios of various sizes. The optimal weights are found from the formula that is discussed in Lecture 6. They are found by taking the ratio of beta to residual variance for asset *j* and dividing this by the sum of ratio of beta squared divided by residual variance across all *i*. That is,

$$\omega_j = \frac{\phi}{\left[\sum_{i=1}^{n} \frac{\beta_i^2}{\sigma_{\varepsilon_i}^2}\right]} \frac{\beta_j}{\sigma_{\varepsilon_j}^2} \qquad 13$$

where ϕ is set equal to 1 under the assumption that the investor seeks to earn an expected return equal to the market.

Part of the interest we have in Table 1 above is to determine how efficient is the optimal portfolio and also to assess how this efficiency changes as the number of assets (mutual funds) grows. The first row in Table 1 is the market variance. Market variance is computed from the average return across all mutual funds. The market variance is the variance of this average mutual fund return (11.3%) across the last ten years. (Note that the average is low because the set of mutual funds includes corporate and government bond funds in addition to common stock funds. The average annual value weighted return to NYSE-AMEX stocks was 14% over the same ten year period, with a standard deviation of 13%.)

Portfolio variance is equal to:

$$V(r_p) = \sum_{i=1}^{n} \omega_i^2 \sigma_{\varepsilon_i}^2 + \sigma_m^2 \phi^2 \qquad 14$$

When ϕ is set equal to 1, portfolio variance is equal to the market variance plus the variance attributable to the residual variances of the assets in the portfolio. This term, $\sum \omega_i^2 \sigma_{\varepsilon_i}^2$, is the inefficiency of the portfolio. It represents the increase in variance of return that could be eliminated if the investor were able to hold a perfectly diversified portfolio.

Table 1 shows the inefficiencies of portfolios of sizes 15, 25, 45, and 85. These portfolios are constructed from the mutual funds that had the highest return (regardless of risk) over the past ten years. Inefficiency falls from 50% to 12% as the number of assets held in the portfolio increases.

Table 1 also shows sub-optimal portfolio construction. The sub-optimal portfolios were constructed from the same assets as the optimal portfolios. However, the weights were set equal across all assets held in the portfolios. Equally weighted portfolios become more efficient as the number of assets increases. However, there is always a substantial increase in portfolio variance due to inefficient portfolio balancing.

One issue that we have omitted thusfar in our discussion of portfolio balancing is the question of which assets to choose for inclusion in a portfolio. Our CAPM model sheds some light on this issue. If we combine equations (13) and (14) above we see that portfolio variance can be written as:

$$V(r_p) = \phi^2 \left[\frac{1}{\sum_{i=1}^{n} \beta_i^2 / \sigma_{\varepsilon_i}^2} + \sigma_m^2 \right] \qquad 15$$

The implications for asset choice are clear. Pick assets with high ratios of beta squared to residual variance. Also note from (15) that as the number of assets in the portfolio grows the sum of the ratio is ever increasing. As *n* approaches infinity, the inefficiency of portfolio variance goes to zero.

Inefficient portfolio variance can be interpreted in terms of lost return by choosing ϕ in a way that equates portfolio variance with market variance. In other words, we have been assuming that the investor is shooting for a target return equal to that expected from the market. Inefficiency in portfolio construction means that the investor bears more risk than would be the case in a transactions-costless world. In contrast, we can ask the question, how much return will the investor lose off of the market return if the investor refuses to bear the additional risk. To get at this we can force equation (15) to equal market variance:

$$V(r_p) = \phi^2 \left[\frac{1}{\sum_{i=1}^{n} \beta_i^2 / \sigma_{\varepsilon_i}^2} + \sigma_m^2 \right] = \sigma_m^2 \qquad 16$$

Collecting terms and simplifying gives:

$$\hat{\phi} = \left[\frac{1}{\dfrac{1}{\sigma_m^2 \sum_{i=1}^{n} \beta_i^2 / \sigma_{\varepsilon_i}^2} + 1} \right]^{1/2} \tag{17}$$

We relabel ϕ as $\hat{\phi}$ to signify that the investor is changing the expectation of return in order to bear no more than the market level of risk. The ratio term

$$\frac{1}{\sigma_m^2 \sum_{i=1}^{n} \beta_i^2 / \sigma_{\varepsilon_i}^2}$$

is the portfolio inefficiency shown in Table 1. That is, under the assumption that ϕ is equal to 1,

$$\frac{1}{\sigma_m^2 \sum_{i=1}^{n} \beta_i^2 / \sigma_{\varepsilon_i}^2} = \frac{\sum \omega_i^2 \sigma_{\varepsilon_i}^2}{\sigma_m^2}$$

As an example, given the portfolio of 15 assets, the inefficiency is 51% and the $\hat{\phi}$ that yields variance equal to the market variance is

$$\left[\frac{1}{.5156 + 1} \right]^{1/2} = .81$$

Recall what ϕ stands for:

$$\phi = \frac{k - r_f}{r_m - r_f}$$

where k is the expected portfolio return. If we set $\hat{\phi}$ = .8, r_m = .12, and the risk free return equal to .05, then k = .106. This means that inefficient portfolio diversification costs the investor 1.4% in expected return if the investor is unwilling to bear more risk than the market level. Note that when the portfolio inefficiency is reduced to 12%, $\hat{\phi}$ goes to .945 and this return shortfall declines to around a quarter of a point.

Finally we turn to the question of portfolio rebalancing. As asset values change, portfolio composition changes and the new portfolio weights are not necessarily optimal. For instance, if one asset increases in value because of a change in formation that has no effect on beta or residual variance, then the default portfolio holdings of that asset after the value increase are too large. The diversifying investor needs to sell off some of this asset and buy more of the others.[5] Portfolio rebalancing involves costly transactions, however, and some marginal analysis of the costs and benefits of rebalancing is appropriate. We can get a clue about the marginal value of rebalancing from equation (17) above.

In equation (17) above, $\hat{\phi}$ tells us the proportion of the risk premium (r_m - r_f) that the investor earns as a consequence of portfolio inefficiency; (1-$\hat{\phi}$) is the proportional cost of inefficiency. Rewritten in terms of the weights $\hat{\phi}$ can be expressed as:

$$\hat{\phi} = \left[\frac{1}{\frac{\sum_{i=1}^{n} \omega_i^2 \sigma_{\varepsilon_i}^2}{\sigma_m^2} + 1} \right]^{1/2} \qquad 18$$

Let's define deviations from the optimal weights in absolute value as $\hat{\omega}_i$. Differentiating $\hat{\phi}$ with respect to one of the weights we have:

$$\frac{\partial \hat{\phi}}{\partial \hat{\omega}_i} = -\hat{\phi}^3 \frac{\omega_i \sigma_{\varepsilon_i}^2}{\sigma_m^2} \qquad 19$$

Substituting from (13) for ω_i we have:

$$\frac{\partial \phi}{\partial \hat{\omega}_i} = -\hat{\phi}^3 \frac{\hat{\phi}}{\left[\sum_{i=1}^{n} \frac{\beta_i^2}{\sigma_{\varepsilon_i}^2} \right]} \frac{\beta_i}{\sigma_{\varepsilon_i}^2} \frac{\sigma_{\varepsilon_i}^2}{\sigma_m^2} = -\hat{\phi}^4 \frac{1}{\sigma_m^2 \sum_{i=1}^{n} \frac{\beta_i^2}{\sigma_{\varepsilon_i}^2}} \beta_i \qquad 20$$

where $\frac{1}{\sigma_m^2 \sum_{i=1}^{n} \frac{\beta_i^2}{\sigma_{\varepsilon_i}^2}}$ is the portfolio inefficiency. So we have:

$$\frac{\partial \hat{\phi}}{\partial \hat{\omega}_i} = -\hat{\phi}^2 (1 - \hat{\phi}^2) \beta_i \qquad 21$$

[5] Harold Mulherin and I explain stock splits on this basis. Stock splits usually occur following a run up in asset value. We argue that splitting is an optimal way for the firm to allow its diversifying shareholders to sell off a portion of their interest in the firm.

Plugging in some real numbers, if the portfolio inefficiency is 12%, then the change in $\hat{\phi}$ for a {.1} change in an individual weight on an asset with a beta of one is .095%. Investment bankers talk about hundredths of percentage points as basis points. The commission on an average trade for them is around 80 basis points. Equation (21) suggests that if eight assets in a portfolio (with average betas of one) were off by .1 each, its probably time to rebalance.

Table 2:

Comparisons of Portfolios of Common Stocks

Average Annual Return on the Market	8.06%
Standard Deviation of Annual Market Return over 5 yrs	12.00%
Average Annual Return on Equally Weighted Portfolio	10.78%
Standard Deviation of Annual Return over 5 yrs on Equally Weighted Portfolio	19.93%
Average Annual Return on Beta Weighted Portfolio	10.54%
Standard Deviation of Annual Return over 5 yrs on Beta Weighted Portfolio	17.55%
Standard Deviation across the 5 yr average Annual Return on Equally Weighted Portfolio	3.47%
Standard Deviation across the 5 yr average Annual Return on Beta Weighted Portfolio	3.23%
5yr Compound Return No Rebalancing	52.96%
Standard Deviation across Portfolios	22.02%
5 yr Compound Return Annual Rebalancing	57.88%
Standard Deviation across Portfolios	22.80%

One hundred NYSE-AMEX stocks with highest ratio of beta to standard error; broken into ten portfolios. Market model estimated from monthly returns over 5 years, 85-89; returns calculated over 90-94.

APPENDIX:

The variance of a portfolio of *n* assets can be written as:

$$
\begin{aligned}
V(r_p) &= E[r_p - E(r_p)]^2 \\
&= E\left[\sum_{i=1}^{n} \omega_i r_i - E\left(\sum_{i=1}^{n} \omega_i r_i\right)\right]^2 \\
&= E\left[\sum_{i=1}^{n} \omega_i r_i - \sum_{i=1}^{n} \omega_i E(r_i)\right]^2 \\
&= E\left[\sum_{i=1}^{n} \omega_i [r_i - E(r_i)]\right]^2
\end{aligned}
$$

Using the CAPM model to define the return to the i^{th} asset gives:

$$
\begin{aligned}
[r_i - E(r_i)] &= \alpha_i + \beta_i r_m + \varepsilon_i - \alpha_i - \beta_i E(r_m) \\
&= \beta_i [r_m - E(r_m)] + \varepsilon_i
\end{aligned}
$$

Substituting into the portfolio variance expression, we have:

$$
\begin{aligned}
V(r_p) &= E\left[\sum_{i=1}^{n} \omega_i (\beta_i [r_m - E(r_m)] + \varepsilon_i)\right]^2 \\
&= E\left[[r_m - E(r_m)]\sum \omega_i \beta_i + \sum \omega_i \varepsilon_i\right]^2 \\
&= \left(\sum \omega_i \beta_i\right)^2 V(r_m) + \sum \omega_i^2 V(\varepsilon_i) \\
&\quad + \sum_i \sum_{j \neq i} \omega_i \beta_i \omega_j Cov(r_m, \varepsilon_j) + \sum_i \sum_{j \neq i} \omega_i \omega_j Cov(\varepsilon_i, \varepsilon_j) \\
&= \left(\sum \omega_i \beta_i\right)^2 V(r_m) + \sum \omega_i^2 V(\varepsilon_i)
\end{aligned}
$$

Since the covariance of the market return with the residual variation of every asset is assumed to be zero as is the covariance between the residual variation of all individual assets, the portfolio variance is only a function of the market variance and the residual variation of each asset. This is the basic principle of the CAPM. Assets are priced in relation to one another based on their covariance with each other, which is captured in beta.