

### OPTION PRICING

Options are contingency contracts that specify payoffs if stock prices reach specified levels. A **call** option is the right to buy a stock at a specified price,  $X$ , called the strike price. (When you buy a call, you buy the right to purchase a stock at a specific strike price; if you sell a call, you sell the right or contract to deliver a stock at a specific price.) In a simple scenario, if a speculator knows that a stock is going to unexpectedly increase in value, the speculator buys a call with a strike price lower than the target level that the stock price will reach. When the stock price increases, the speculator exercises the call by buying the stock at  $X$  and selling it at its new price, which is higher than  $X$ . A **put** option works in opposite fashion. A put is the right to sell a stock at a specified price. If one believes that a stock price will decline, a put option is a way of capitalizing on this information, i.e., buy a put which gives the right to sell the stock at  $X$ . When the price falls, buy the stock on the open market and use these shares to exercise the put.

The theoretical price of options was developed by Fisher Black and Myron Scholes. Their model is complicated mathematically, but the general principles of option pricing are fairly straightforward. Let's start with the simplest comparative statics.

- 1) Assume that the expected value of a stock at time  $T$ ,  $E(S_T)$ , is \$100. Assume that the exercise price,  $X$ , of a call option is \$90. Then we expect that the option price should be somewhere around \$10. This is because if you buy the call you get the right to buy an asset with an expected value of \$100 for \$90. Profit maximizing traders would pay something around \$10 for the call option because this is an approximation of its expected value.
- 2) Following this line of reasoning, if  $X$  decreases, the call price,  $C$ , increases. Holding  $E(S_T)$  at \$100, if  $X$  goes to \$80, then the expected value of the call option is approximately \$20.
- 3) The current price of the stock, call this simply  $S$ , forms the basis for the expected value at time  $T$ ,  $E(S_T)$ . That is, if  $S$  increases, so too does  $E(S_T)$ . Hence, if  $S$  increases, holding  $X$  constant, then the call price,  $C$ , has to go up.
- 4) If the variance of the distribution of stock prices goes up, then the call price has to go up. The reason can be seen by considering the case where  $E(S_T) = \$100$  and  $X = \$100$ . The call option still has value and will have a positive price,  $C$ , because there is some chance that the realized price at  $T$  will be greater than  $X$ . Traders are willing to pay something for this potential profit. Obviously then, as the variance of the distribution of the stock price goes up, it increases the probability that the stock price will be greater than  $X$  at time  $T$ . Hence, traders will increase the amount,  $C$ , they are willing to pay for the option.

The option price is also a function of the time to maturity,  $T$ , and a function of the risk free interest rate. These relations are a bit more complicated. The effect of time to maturity is clear enough for American options but less so for European options. American options have the characteristic that they can be exercised at any point. That is, if you have a call option to buy a stock at  $X$ , you can exercise that option anytime before  $T$  and buy the stock. Hence, if the stock price goes above  $X$ , you can exercise at a profit. The longer is  $T$ , the greater the chance that the stock price will bounce above  $X$  at some point. This means that the call price will be a positive function of the time to maturity.

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However, with European options, a call gives you the right to buy the stock at time  $T$ , not anytime before  $T$ . Based on this, it is not as clear that  $C$  and  $T$  are positively related.

- 5) However, when we consider that  $E(S_T)$  is expected to grow at a rate determined by its relative risk (this is the expected return based on CAPM), then we can say even for European options,  $C$  is positively related to  $T$  holding  $X$  constant.
- 6) Finally, based on the same logic, the call price should be positively related to the risk free rate of interest because the expected return on the stock and hence the expected stock price at  $T$  are based on the risk free through CAPM.

### The Black-Scholes Model

These are the general principles of option pricing. The precise derivation of option pricing comes from Black and Scholes. Their insight into option pricing comes from considering a trading strategy that involves selling a stock short and then buying a call to cover the short sale. This is called a **short-sale, call-covered hedge** strategy.

A short sale is the sale of borrowed shares of a stock. Certain investors, typically pension funds that engage in buy and hold portfolio management, are willing to lend shares of stock from their portfolio for the purpose of short sales. The lender of shares retains the right to recall the shares and is promised to be paid any dividends or distributions that are paid on the shares during the time they are lent out. Lenders are typically paid a nominal fee for lending their shares. It doesn't cost much to lend out shares under this arrangement, and therefore they aren't paid much.

The investor who borrows the shares and then sells them short is obligated to replace the shares at some point. It is this replacement that is hedged by the purchase of a call. The investor covering the shorted share with a call has contracted directly for the repurchase of shares to cover the replacement of the borrowed shares.

The call-covered, short-sale strategy has the following cash flows. The investor receives  $S$  from the short sale of the stock. The call option that covers this position costs  $C$ . Thus, the investor has net cash of  $(S - C)$  that can be invested in other assets.

### The Riskless Short-Sale Hedge

It is enlightening to consider how this strategy plays out in the case where there is no risk. That is, assume that the price of the stock increases at the rate  $r$  without variance, i.e.,

$$S_t = Se^{rt}$$

Further, assume that the proceeds of the covered short sale  $(S - C)$  are invested in the risk free asset and earn  $r_f$ .

Thus, the investment strategy returns as amount

$$Y = (S - C)e^{r_f T}$$

at the expiration of the option at time  $T$ . From this it is clear that if money made by the investment strategy,  $Y$ , is greater than the exercise price of the option,  $X$ , the investor has made excess profits. On the other hand, if  $Y < X$ , the investment strategy has lost money. Competition will force excess profits and losses to zero. Hence,  $Y = X$ , which means that

$$(S - C)e^{r_f T} = X$$

Rewriting makes the expression more meaningful:

$$C = S - Xe^{-r_f T} \tag{1}$$

Competition will force the price of the call,  $C$ , to adjust until excess profits and losses are zero. In the case of zero variance in the stock price, the call price soaks up the difference between the current stock price and the riskfree discount of the future strike price on the call option. The equilibrium is achieved by adjustments in  $C$ .

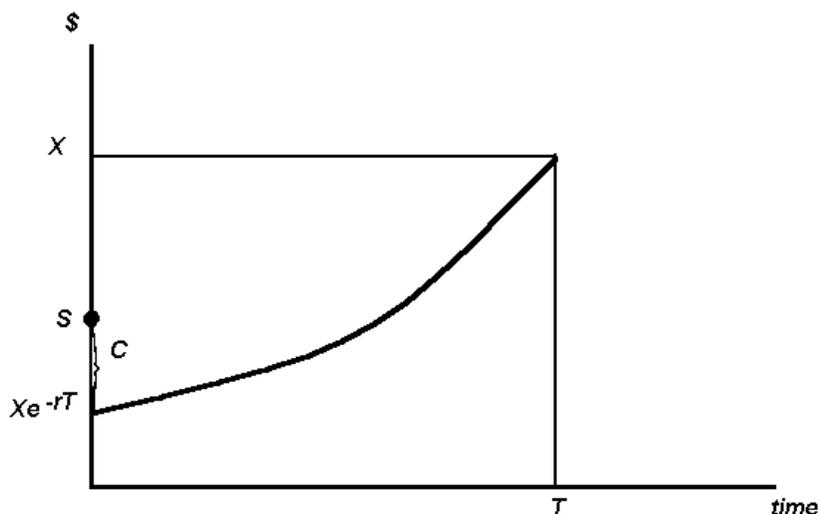
We can describe the competitive equilibrium in terms of the parameters in the problem, and the simple formula (1) reveals most of the comparative statics of the problem. From it we see directly that

$$\frac{\partial C}{\partial S} > 0, \quad \frac{\partial C}{\partial X} < 0, \quad \frac{\partial C}{\partial r_f} > 0, \quad \text{and} \quad \frac{\partial C}{\partial T} > 0.$$

These results are identical to the comparative statics that we developed more intuitively in the first section.

A graph is illustrative:

Figure 1



In Figure 1 we see that  $S - C$  are the net proceeds of the short sale. Invested in the risk free asset these proceeds return some value  $Y$  after  $T$  periods. If  $Y$  is greater than  $X$ , the portfolio makes excess returns; if less, negative returns; if  $Y = X$ , the portfolio earns zero excess. The proceeds of the investment in the riskfree asset are exactly equal to the cost of redeeming the call. The discounted value of  $Y=X$  is  $Xe^{-r_f T}$ , which is equal to  $S-C$ .

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If  $C$  falls, then the investment returns an excess; if it grows, a shortfall. Graphically, we see that as  $S$  increases, so must  $C$ . If  $X$  decreases,  $C$  increases in order to maintain the equilibrium. The effect of changes in the risk free rate and the period of execution have effects that are also obvious. Notice that in this model where there is no random variation in the stock price, the stock price and call price vary in 1 to 1 fashion.

### The Risky World

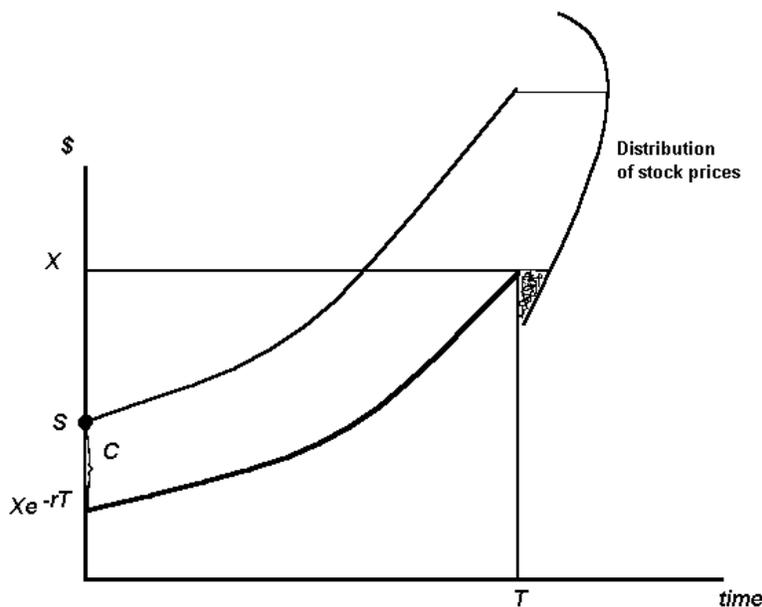
Now let's consider what happens when the stock price has a random component. Figure 2 shows what happens when  $S$  has a distribution. The stock price increases at some compound rate. CAPM tells us that the expected value of the stock price is give by

$$S_T = Se^{rT}$$

where  $r$  is the risk adjusted return predicted by the stock's beta. However, the expected return is not certain. The actual return on the stock varies across time as shown by the market model. The variation is given by the error term in the market model the standard deviation of which we can call  $s$ .

Figure 2 shows the density, shown by the curved line to the right of graph, that describes the end points of the various paths that the stock price may take. Variation in the stock price has no effect on the hedge portfolio except in the case where the stock price varies into the lower tail of the distribution. If the stock price falls into the shaded part of the density, the short sale position can be covered, without exercising the call option, at a price less than  $X$ . This means that the hedge portfolio makes more money than the risk free rate.

Figure 2



This means that in the case where the stock price has variation, if  $C = S - Xe^{-rT}$ , the short-hedge portfolio makes excess profits. Competition cannot stand excess profits. As a result,

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$C$  must be larger than the discounted value of the short sale/call option cover. That is, in the risky world,

$$C > S - Xe^{-r_f T}$$

In fact, the competitively priced difference

$$OV = C - [S - Xe^{-r_f T}]$$

is the option value of the call security. It represents the competitive value of having an opportunity to do something (buy the stock at the price  $X$ ) even if it turns out that you don't do it.

Competition drives the price of the call up so that there are no excess returns, risk adjusted. In doing so, the variation between the call price and stock price is no longer 1/1. Now, the call price varies less than the stock price because as the stock price goes up, the shaded area in Figure 2 becomes smaller. In the no-variance problem an increase in the stock price had to be offset by an increase in the call price. Now the call price embodies some expectation that variation in the stock price will allow the stock to be repurchased below the exercise price of the call. As the stock price goes up, this part of the call price goes down, not up. Hence, the relation between the call price and stock price is positive but less than one to one.

The comparative static result with respect to  $\sigma$  is clear. As the variance of the stock return increases, the shaded area in Figure 2 increases and the call price goes up. This is the last comparative static result:

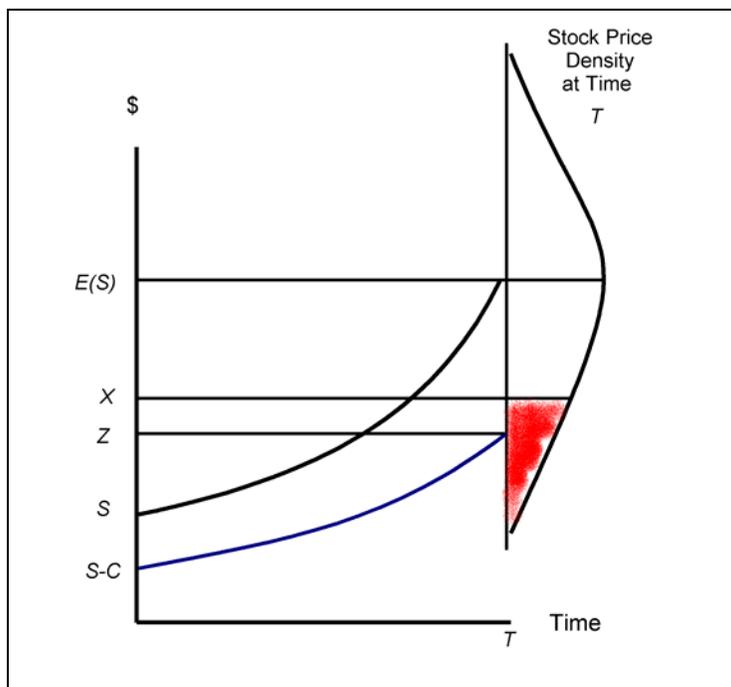
$$\frac{\partial C}{\partial \sigma} > 0$$

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The actual competitive equilibrium price of the call can be deduced from Figure 2. Imagine a call price,  $C$ , such that the proceeds of the call covered short sale,  $S-C$ , lies below  $Xe^{-r_f T}$ . Now, project this line out to time  $T$ . Competitive equilibrium option pricing requires that this line hits the shaded area of the stock price distribution (the portion below  $X$ ) at its expected value.

This is shown in Figure 3. There  $C$  is set so that the line starting at  $S-C$  grows at the risk-free rate for  $T$  periods at which time it is equal to a value labeled  $Z$ .  $Z$  is the expected value of the stock price below  $X$ , i.e., the middle of the probability weighted values of  $S_T$ . If the actual stock price is above  $Z$  but below  $X$  then the hedge strategy loses money. If the actual price is below  $Z$  the strategy makes excess profits. At  $Z$ , excess profits and losses are zero.

Figure 3: Option Pricing



Black and Scholes derive the formula that draws this line. Their formula is:

$$C = S \cdot N\left(\frac{\ln(S / X) + [r_f + (\sigma^2 / 2)]T}{\sigma\sqrt{T}}\right) - e^{-r_f T} \cdot X \cdot N\left(\frac{\ln(S / X) + [r_f - (\sigma^2 / 2)]T}{\sigma\sqrt{T}}\right)$$

where  $N(\cdot)$  is the cumulative unit-normal distribution. The formula is not that complicated even if it is difficult to precisely derive. The formula is based on the short-sale hedge formula (1), where each term is weighted by a probability which captures the expected value of the stock price density below  $X$ .

The Black-Scholes formula can be implemented using any standard spreadsheet program. The function for  $N(\cdot)$  in Excel is NORMSDIST(.). Let's assume that  $S$  is \$1,  $X$  is \$1,  $r_f$  is 5%, and  $T$  is 5 years. Sigma is the standard deviation of the return on stock. This can be estimated from historical data. For instance, the range of the standard deviations of the daily returns on the firms that we look at in the Challenger crash are:

Lockheed	.026
Rockwell	.017

These translate into annual standard deviations of .41 and .27, respectively. (The units of measure of  $s$  and  $T$  have to be the same.)

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Given the parameters of  $S = \$1$ ,  $X = \$1$ ,  $r_f = 5\%$ , and  $T = 5$ , the B-S formula gives the following:

$\sigma$	$C$
.5	50¢
.8	67¢
.4	43¢

Alternatively, assume  $S = \$100$ ,  $r_f = .05$ , and  $s = .41$ . Some other simulations yield the following:

$X$	$T$	$C$
\$100	1	\$18.54
\$120	1	\$11.34
\$140	1	\$ 6.82
\$120	2	\$19.93

### Applications

The Black-Scholes option pricing formula is used a good bit in the finance industry. Some people attempt to trade options based on this formula. However, that is not particularly profitable because the formula is only an approximation. The formula is based on European style options that can only be exercised at time  $T$ . American options can be exercised at any time up to  $T$ . Also, the formula is in error if the distribution of returns is not normal. This seems to be the case for deep in and out of the money options. Finally, the formula require two inputs,  $s$  and  $r_f$  that are not known perfectly. This is especially true for  $s$ .

It is unlikely that options are used often for call-covered, short-sale hedge investments. However, since they could be, that strategy is a way of defining their value. The Black-Scholes model should be thought of as a thought experiment. It tells us how European-style options would be priced with full knowledge of the parameters and in the absence of transactions costs in executing the call-covered short-sale. Obviously, transactions costs exist, but nonetheless the B-S model gives the equilibrium value of the option securities.

One interesting application of the B-S model is the derivation of implied volatility. A derived value of  $s$  can be constructed from the observed values of the option price and the other values. This value of  $s$  is called implied volatility. Implied volatility measures are tracked for the S&P 100 and for treasury futures options.<sup>1</sup> It has been shown that implied volatility has more predictive power than historical volatility.

Notes:

Implied volatility is the *calculated* value of the standard deviation of the stock's return using the B-S model. Implied volatility is a method for *estimating* the variance of the expected return on the stock. The estimator is the B-S formula.

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<sup>1</sup> See Bertonazzi & Maloney, Journal of Fixed Income, December 2001.

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The value of a call has to be tied to the potential profit available from engaging in a call covered short sale. So, more intuitively, the short sale proceeds make money at the risk free rate. If the risk free increases the value of the call covered short sale increases and hence the value of the call must increase to capitalize this gain.

Possibly more intuitively, the call is an option to claim to something in the future. If the value of that something increases, the value of the claim increases.

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### Stock Option Pricing—Addendum (2/5/04)

Someone yesterday asked the question, “How well do stock options predict stock prices?” I did not give a responsive answer. Here is a better discussion.

The problem is that stock options are not written in a way that make a clear prediction of future stock prices. For instance, if a stock is currently trading for \$100, there will be call options with strike prices of \$95, \$100, \$105, \$110, etc. In other words, there is no explicit prediction of future stock price, merely break points beyond which the option is valuably exercised.

That is why the idea of the call-covered, short-sale hedge strategy is informative. Efficiency requires that the call price adjust until the expected value of this investment strategy is zero.

And, the Black-Scholes formula gives us a call price that makes this strategy have a zero expected value. The formula is derived using the assumption that the distribution of stock prices is normal.

In the large, or more precisely, for options with strike prices close to the current price of the stock, the B-S formula predicts well. This suggests that option prices are efficiently priced relative to the underlying stock.

For options with strike prices far away from the current stock price, the B-S formula gives prices that are not close to the observed prices. This means that (1) either option prices are irrationally priced, or (2) the B-S formula is imperfect.

Because the B-S formula is so close for near the money options, we tend to think that (2) is more likely than (1) for away from the money options.

Traders use the B-S formula, but if options are trading at prices that are not equal to the B-S formula, one cannot count on making money by arbitraging according to the formula.

The formula includes two parameters that are not perfectly known: the risk free return and the variance of the stock return.

### Call-Covered Short Sale

Short sell a stock for price  $S_0$ ; by a call with a strike price,  $X$ , that matures at  $T$  and costs  $C$ .

The payoff to this hedged portfolio<sup>2</sup> at time  $T$  is:

$$[S_0 - C]e^{iT} - E[\min(X, S_T)]$$

The investor earns interest on the proceeds from the short sale less the cost of the call during the holding period. Then the investor must cover the short position in the cheapest possible way. If the stock is trading at or below  $X$  when it is time to cover, the investor buys the stock in the open market. However, if the stock price at  $T$  is greater than  $X$ , the investor exercises the call option and covers the short position that way.

The call-covered short-sale portfolio defines a pricing equilibrium. The payoff must equal zero:

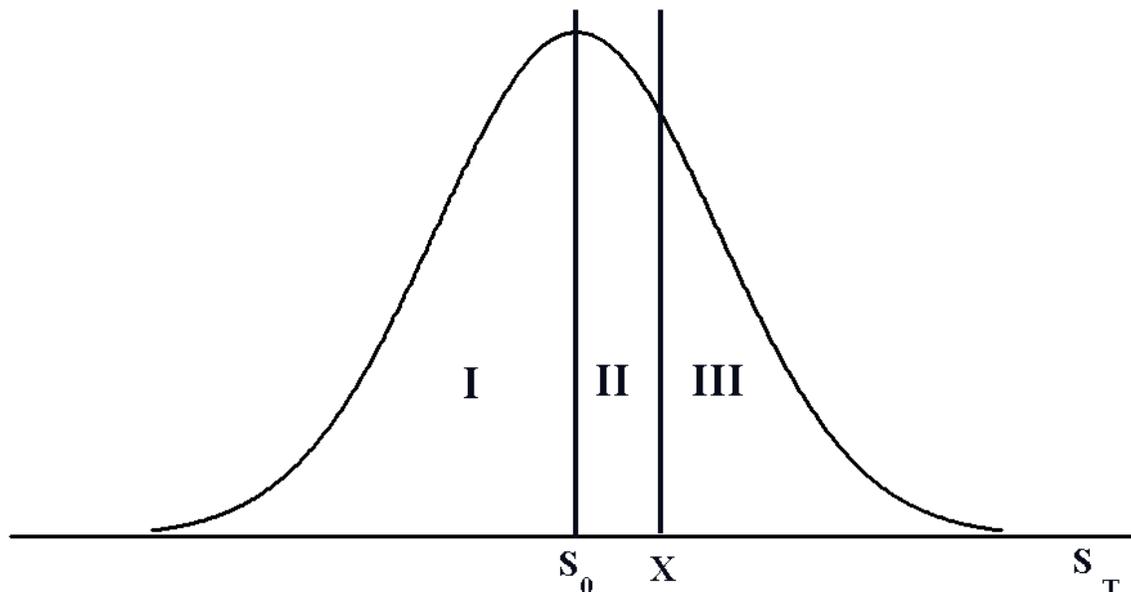
$$[S_0 - C]e^{iT} - E[\min(X, S_T)] = 0$$

and this zero-profit equilibrium is achieved by changes in  $C$ . The option is priced so that there are no expected excess profits from forming such portfolios.

To understand this equilibrium, we must explore the expected value:  $E[\min(X, S_T)]$ . To do this, consider the following graph. It shows the distribution of possible stock prices at time  $T$ . The distribution is normal and centered on the current stock price. Black & Scholes assume that there is no expected appreciation in the stock price. They do this to simplify the math of their derivation. More recent pricing models have accommodated drift in the stock price, but they do not add much. The graph identifies three areas: I:  $[S_T < S_0]$ , II:  $[S_0 < S_T < X]$ , and III:  $[X < S_T]$ . The area III identifies the circumstance in which the investor exercises the call option to cover the short stock position.

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<sup>2</sup> The portfolio is hedged because the call protects the investor from run-up in the stock price.



Notice that in area I, ignoring the cost of the call, the investor makes money. In areas II and III, the investor loses money. However, in area III, the investor does not lose as much money as would be the case if the short position were not hedged with the call. The losses in area III are capped at  $X$  instead of being potentially unlimited.

The expected value of the cost of covering the short position at  $T$  is just the integral of the payoffs in the three areas times the density function. We can write this as follows:

$$E[\min(X, S_T)] = \int_{-\infty}^{S_0} s \cdot n(s; S_0, \sigma^2) ds + \int_{S_0}^X s \cdot n(s; S_0, \sigma^2) ds + \int_X^{+\infty} X \cdot n(s; S_0, \sigma^2) ds$$

where  $s$  is the value of the stock at  $T$ , and  $n(\cdot)$  is the normal density with mean  $S_0$  and standard deviation  $\sigma$ . So the three different integrals are expected values over the ranges of the distribution. They sum to the expected cost of covering the short position.

We can rewrite the equilibrium condition in terms of the call price:

$$C = S_0 - E[\min(X, S_T)]e^{-iT}$$

The following comparative static properties of the option price equilibrium can be derived:

$\frac{\partial C}{\partial S_0} = 1$ : As the expected stock price increases the call must increase to offset the value of the short-sale proceeds.

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$-1 < \frac{\partial C}{\partial X} < 0$ : As the strike price goes up, the call price falls because the expected value in area III is declining. However, this is not 1:1.

$\frac{\partial C}{\partial T} > 0$ : As the time to maturity increases, the value of the option increases. While we intuitively think this because the longer the maturity the more option we have, that only applies to American options. This framework identifies the pricing equilibrium for European options that can only be exercised on day  $T$ . Hence all of the effect of  $T$  on  $C$  comes from the interest earned on the short-sale proceeds.

$\frac{\partial C}{\partial i} > 0$ : Ditto above. The interest rate has a position effect on  $C$  because of the short-sale earnings. We use the risk free because in an expected value since, this is a risk free investment strategy.

$\frac{\partial C}{\partial \sigma} > 0$ : This is the big one. The volatility of the stock price is positively related to the option price. The reason is because as the standard deviation of the distribution of the stock price increases, the losses avoided by the hedged position for stock price increases grow. In other words as the distribution of the stock price spreads out, the gains in area I grow faster than the losses in areas II and III, and this is because of the hedge created by the option. Hence, the call price has to increase to offset these expected profits.

The SAS program below generates areas I, II, and III and calculates the expected value of the cost of covering the short sale.

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BS Simulation:

```
data bs;
So=100; mu=.0; sigma=.1386; X=105;
do i=1 to 100000; drop i;
z=rannor(1234567); p=probnorm(z)-probnorm(z-1/28182);
St=So*(1+(mu+sigma*z));
pSt=p*min(X,St); dpSt=pSt*exp(-.05);
output; keep p mu sigma z St So pSt X dpSt; end;
proc means sum mean std; var p pSt dpSt z sigma St; run;
proc means sum; var p dpSt pSt;
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where  $z < 0$ ; run;

proc means sum; var p dpSt pSt;

where  $St < X$  and  $St > So$ ; run;

proc means sum; var p dpSt pSt;

where  $St > X$  ; run;