

PRICE THEORY**Michael T. Maloney**

The methodology of economics is the object of continual debate. We are not going to engage in that debate directly, though implicitly we will live in the middle of it. The criteria by which we will choose topics for study are based on the power of analytical models to make accurate predictions. Our chore is to learn how models are constructed to achieve this end.

The practice of economic science is based on two cornerstones. The first is the self-interested or *optimizing* behavior of individual economic agents. Individuals are assumed to optimize an objective function subject to exogenous constraints. Their behavior is described by the principles of optimization and formulated as a function of the exogenous, parametric constraints that they face.

The second cornerstone of economics is the equilibration of the offsetting behavior of individual economic agents. This process of market equilibrium makes some of the parametric constraints faced by individuals become endogenous to the model. The equilibrium variables then become functions of other parameters of the model.

The scientific content of economic models is that changes in the values of the choice variables of individuals and equilibrium variables in markets are predicted based on changes in the exogenous parameters. The simplest example is the model of Supply and Demand.

Part of our effort is to learn to use some tools of analysis.

■ SOME SIMPLE ALGEBRA¹

As an example of Comparative Static Analysis, consider a linear supply and demand model:

$$Q_D = b_0 + b_1 P, \text{ where } b_1 < 0 \quad 1$$

and

$$Q_S = c_0 + c_1 P, \text{ where } c_1 > 0 \quad 2$$

The equilibrium condition is that:

$$Q_D = Q_S \quad 3$$

Solving demand and supply simultaneously based on the equilibrium condition gives:

$$b_0 + b_1 P = c_0 + c_1 P \quad 4$$

or

¹ Primary material in Silberberg (1990)&(2001) Sections 1.4 and 1.5. examples with comparative statics.

$$P^* = \frac{c_0 - b_0}{b_1 - c_1} \quad 5$$

- Note the use of the *star* notation. In equation (5) P^* is the solution value of the interaction of the behavior functions described by equations (1) and (2). Equations (1) and (2) describe Q as a function of the parameters b_0 , b_1 , c_0 , c_1 , and P . For instance, in the demand equation, theory says that consumer behavior in terms of the choice of the level of consumption is determined by price. Price is an exogenous constraint. Similarly we can describe quantity supplied as exogenously determined by price. Equations (1) and (2) describe optimization behavior on the part of economic agents. Equation (5) expands the analysis to include the market equilibrium interaction of the behavior of economic agents. In this process P changes from a parameter to a variable that is a function of the other, demand and supply parameters of the model. P^* is a function and is different than the parameter P . The star notation says: "This is something new. The result of economic behavior."

- Comparative Static Analysis is nothing more than the examination of the function P^* . This is accomplished easily. The function is differentiated with respect to (w.r.t.) its arguments. For instance, if we want to know how equilibrium price responds to an outward shift in demand, we differentiate P^* w.r.t. b_0 .

$$\frac{dP^*}{db_0} = \frac{1}{c_1 - b_1} > 0 \quad 6$$

The comparative static answer is that dP^*/db_0 is positive. We know this because we have asserted that c_1 is positive and b_1 is negative.

- SOME USEFUL DERIVATIVES and derivations

The chain rule is an important trick to remember. For $y = f(g(x))$, let $u = g(x)$, so that $dy/dx = dy/du \cdot du/dx$. As an application consider a demand and supply model identified for exponential functions where we have:

$$Q_D = b_0 P^{b_1}, \text{ where } b_1 < 0 \quad 7$$

$$Q_S = c_0 P^{c_1}, \text{ where } c_1 > 0 \quad 8$$

Solving demand and supply simultaneously based on the equilibrium condition gives:

$$P^* = \left(\frac{c_0}{b_0} \right)^{\frac{1}{b_1 - c_1}} \quad 9$$

Use the chain rule to take the derivative w.r.t. b_0 . Let $u = c_0/b_0$.

$$\frac{dP^*}{db_0} = \left(\frac{1}{b_1 - c_1} \right) \left(\frac{c_0}{b_0} \right)^{\frac{1}{b_1 - c_1} - 1} \left(\frac{-c_0}{b_0^2} \right) \quad 10$$

• Another thing to keep in mind when dealing with exponential functions is shown in developing the definition of elasticity. Price elasticity is defined as the percentage change in quantity demanded that results from a percentage change in price. In derivative form we have

$$\varepsilon = \frac{dQ}{dP} \frac{P}{Q} \quad 11$$

Consider the exponential demand curve shown in equation (7). The derivative of quantity w.r.t. price looks like

$$\frac{dQ_D}{dP} = b_1 b_0 P^{b_1 - 1} = b_1 \frac{Q}{P} \quad 12$$

The trick is to recognize that Q can be substituted back into the expression for the derivative. Cross multiplying shows that b_1 is elasticity.

• The exponential demand curve is often written in logarithmic form as:

$$\ln Q = b_0 + b_1 \ln P \quad 13$$

This is useful because the functional parameters are linear in $\log P$ and Q . Hence, the functional form is conveniently employed econometrically. Note that the derivative of $\log Q$ w.r.t. $\log P$ is b_1 which we have already defined as elasticity.

• There are three general forms of demand curves used in the practice. These are linear, semi-log, and double log. These are listed below written in inverse demand form. You should be able to graph the demand, marginal revenue, and total revenue functions.

Linear:

$$P = a + bq \quad \text{and} \quad Pq = aq + bq^2 \quad 14$$

$$P' = b, \quad R' = a + 2bq \quad \text{and} \quad R'' = 2b \quad 15$$

Semi-Log:

$$P = \alpha + \beta \ln q \quad \text{and} \quad Pq = [\alpha + \beta \ln q]q \quad 16$$

$$P' = \frac{\beta}{q}, \quad R' = P + \beta \quad \text{and} \quad R'' = \frac{\beta}{q} \quad 17$$

Double-Log:

$$P = Aq^B \quad \text{and} \quad Pq = Aq^{B+1} \quad 18$$

$$P' = BAq^{B-1}, \quad R' = (B+1)Aq^B \quad \text{and} \quad R'' = B(B+1)Aq^{B-1} \quad 19$$

■ PROFIT Maximization

The model of profit maximization offers a vehicle for the discussion of the simple optimization problem. We commonly separate the model of profit maximization into two sections. One deals with profit max for the competitive firm and the other for the monopoly.

For the competitive firm, price is a constant. Profit is defined as the difference between revenue and cost where revenue is the constant price time quantity and cost is described by a function that maps the output choice of the firm into the expenditures that the firm makes on resources in order to produce that level of output. The choice variable of the firm is output, which we label as q . Thus we have:

$$\max_{\{q\}} \pi = Pq - C(q) \quad 20$$

The First Order Condition (FOC) of the problem is:

$$\frac{d\pi}{dq} = P - C'(q) = 0 \quad 21$$

The Sufficient Second Order Condition (SSOC) is:

$$\frac{d^2\pi}{dq^2} = -C''(q) < 0 \quad 22$$

The FOC implicitly defines the value of output, which we will label q^* . It implicitly defines this because given the general statement of the total and marginal cost functions $C(q)$ and $C'(q)$ we cannot solve for q directly in equation (21). By the Implicit Function Theorem, we can be sure that Equation (21) can be solved for the optimizing value q^* if the SSOC hold. Equation (22) says that for a maximum, marginal cost must be rising (the slope of marginal cost, C'' , must be positive).

Of interest to us in this problem is the prediction about the behavior of the firm. So far we have only defined a rule for optimizing behavior. The next and crucial step is to derive an expression that tell us how the firm will change its behavior when the constraint that it faces change. This is the essentially element of a model. It must predict behavior.

In order to derive this prediction we undertake what is called comparative static analysis. We do this by differentiating the FOC of the model evaluated at the optimum value with respect to one of the parameters of the model. In mathematical form when the parameter price changes, the comparative static analysis is given :

$$\frac{d(d\pi/dq)}{dP} = \frac{d[P - C'(q^*)]}{dP}$$

This gives:

$$1 - C''(q^*) \frac{dq^*}{dP} = 0$$

or

$$\frac{dq^*}{dP} = \frac{1}{C''} \quad 23$$

Equation (23) says that firm will increase its output if price goes up. The right-hand side must be positive by the SSOC. Notice that the standard graphical analysis of this problem says exactly the same thing.²

- Next let's examine the behavior of the monopoly firm.

The monopolist maximizes profit according to the same objective function as the competitive firm except that for the monopolist price is not a constant. Rather price is a function of the output decision of the firm. Thus, we write:

$$\max_{\{q\}} \pi = P(q)q - C(q) \quad 24$$

The FOC is:

² Also note that the standard graphical analysis using a U-shaped average and marginal cost curve adds restrictive assumptions to the model that the mathematics do not include. There is nothing wrong with restrictive assumptions. However, it is important to distinguish between what is in the math and what is not there. Also, if price is changed in the graph, notice that the firm responds up the marginal cost curve but the math says that the firm responds according to the inverse of the marginal cost curve. This difference is because the graph is rotated 90 degrees relative to the math.

$$\frac{d\pi}{dq} = P'(q)q + P(q) - C'(q) = 0 \quad 25$$

The Sufficient Second Order Condition (SSOC) is:

$$\frac{d^2\pi}{dq^2} = P''(q)q + 2P' - C''(q) < 0 \quad 26$$

In this model it is sometimes useful to simplify by substituting a general expression of revenue for the price times quantity expression. This gives:

$$\max_{\{q\}} \pi = R(q) - C(q) \quad 27$$

The FOC is:

$$\frac{d\pi}{dq} = R'(q) - C'(q) = 0 \quad 28$$

The Sufficient Second Order Condition (SSOC) is:

$$\frac{d^2\pi}{dq^2} = R''(q) - C''(q) < 0 \quad 29$$

We know that R' is marginal revenue and therefore $P'q+P$ shown in equation (25) is also an expression for marginal revenue.

Deriving the comparative statics for this model is a little more complicated than for the competitive firm. The reason is that there are no parameters explicitly identified in the model. To solve this problem we define a parameter that stands for the change in constraints facing the firm that we wish to analyze. To this end, let α stand for a shift in the revenue function. Thus, we can write the FOC evaluated at the optimum in terms of the q^* and α :

$$R'(q^*, \alpha) - C'(q^*) = 0$$

Differentiating with respect to α gives:

$$[R''(q^*, \alpha) - C''(q^*)] \frac{dq^*}{d\alpha} = - \frac{dR'}{d\alpha}$$

Rewriting, we have:

$$\frac{dq^*}{d\alpha} = - \frac{\frac{dR'}{d\alpha}}{R''(q^*, \alpha) - C''(q^*)}$$

Let's define $\frac{dR'}{d\alpha}$ as positive. That is, let's assume that an increase in α causes the marginal revenue function to shift up.³ If this is true, then the model clearly predicts that an increase in α will cause optimal output to increase. The denominator is negative by the SSOC and this cancel the minus sign out front.

- One of the important conclusions to draw from this analysis is that the predictable differences between monopoly and competitive behavior are few. This is why it is so hard to determine whether a market is monopolized or not. In the case we just investigated, we find that both competitive and monopoly firms are predicted to increase output in the face of an increase in demand. This is a simple result, but similar conclusions hold on virtually every margin of behavior. Whenever you are asked if the monopolist will respond differently from a competitive firm, the starting point of your thinking should be that they will act qualitatively the same.

Also recognize that the model of the monopoly that we have just investigated is more complete than the model of the competitive firm. The model of the competitive firm does not include a discussion of market equilibrium. To get this, we must aggregate the actions of these competitive firms. The monopoly model assumes both a behavior model for the firm and a market clearing relation among demanders.

³ This is purely arbitrary. It is our assumption and thus it can be assumed to be anything we want. The point is we want to know the effect of an increase in the revenue function and this assumption defines that increase as an upward shift in the marginal revenue function. We might want to further investigate what kind of change in the demand function would give this shift.

Bi-Market Equilibria

Let the demand for different products 1 and 2 be $q_1 = 10 - 2p_1 + p_2$ and $q_2 = 10 + p_1 - 2p_2$ where q_i and p_i represent the respective quantity demanded and price of the i th good. These are normal demand functions. Quantity demanded is a function of own price and the price of a substitute good.⁴ The intercept term in these demand functions represents income, tastes, and other prices.

Competitive Case

Assume industry supply for both goods is described by $m_i = 2 + q_i$, where m is the supply price (i.e., marginal cost).

The equilibrium condition is defined by $q_D = q_S$ and p_i equals marginal cost. Thus, we have:

$$p_1 - 2 = 10 - 2p_1 + p_2$$

$$p_2 - 2 = 10 - 2p_2 + p_1$$

In single market equilibrium analysis, we would have:

$$p_1 = \frac{12}{3} + \frac{p_2}{3}$$

To solve for the bi-market equilibrium we must solve the two single market equilibrium expressions simultaneously. Rewriting in matrix form:

$$\begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} -12 \\ -12 \end{bmatrix}$$

Solving by Cramer's Rule gives:

$$p_1 = \frac{\begin{vmatrix} -12 & 1 \\ -12 & -3 \end{vmatrix}}{\begin{vmatrix} -3 & 1 \\ 1 & -3 \end{vmatrix}} = \frac{36 + 12}{9 - 1} = \frac{48}{8} = 6$$

Solving similarly for p_2 gives the same answer as the two markets are symmetric. Substituting price into the demand or supply function gives $q_i = 4$.

Consider the following **comparative static analysis**. In the demand function, let the intercept term change by +1, for instance, due to an increase in income. The result in the single market equilibrium expression is:

⁴ The Theory of Consumer Behavior tells us that quantity demanded is a function of prices and income. Own price generally has a negative effect. The net of all other prices is positive.

$$p_1 = \frac{12+1}{3} + \frac{p_2}{3} = \bar{p}_1 + \frac{1}{3}$$

That is, equilibrium price goes up by one-third.⁵ When we examine the bi-market equilibrium we get:

$$p_1 = \frac{\begin{vmatrix} -12 & -1 & 1 \\ -12 & -1 & -3 \\ -3 & 1 & 1 \\ 1 & -3 & -3 \end{vmatrix}}{\begin{vmatrix} -3 & 1 \\ 1 & -3 \end{vmatrix}} = \frac{36 + 3 + 12 + 1}{9 - 1} = \frac{48 + 4}{8} = 6 + \frac{1}{2}$$

Monopoly

Assume that each market is monopolized separately. The problem is set up like this:

$$\max_{\{q_1\}} \pi^1 = p_1 q_1 - \int m_1 dq_1$$

To solve this problem it is most convenient to invert the demand curves, changing them from the Hicks form [$q=f(p)$] to the Marshallian form. For a single market this gives:

$$p_1 = \frac{10}{2} - \frac{q_1}{2} + \frac{p_2}{2}$$

The FOC is:⁶

$$\frac{\partial \pi^1}{\partial q_1} = p_1 + q_1 \frac{\partial p_1}{\partial q_1} - m_1 = 0$$

Substituting gives:

$$5 - q_1 + \frac{p_2}{2} = 2 + q_1$$

$$q_1 = \frac{3}{2} + \frac{p_2}{4}$$

Similar analysis for market 2 gives:

⁵ Note that we are assuming that the intercept in each demand equation goes up by 1. This would be consistent with an increase in income, which would apply to all demand curves. As a problem, solve for the effect on equilibrium price and quantity in both market that would result from a change in the intercept term of the supply function in one market.

⁶ Note the derivative that gives marginal cost. The derivative of an integral w.r.t. the variable of integration is just the integrand.

$$q_2 = \frac{3}{2} + \frac{p_1}{4}$$

The equilibrium can be solved in a number of ways. The basic idea can be seen by substituting the Hicksian demand curves on the left-hand side of these expressions. This gives:

$$10 - 2p_1 + p_2 = \frac{3}{2} + \frac{p_2}{4}$$

$$10 - 2p_2 + p_1 = \frac{3}{2} + \frac{p_1}{4}$$

The bi-market solution in the case of separately monopolized markets is given by the simultaneous solution to these two equations. Again it is a valuable exercise to think of this problem in matrix terms.

$$\begin{bmatrix} -2 & 3/4 \\ 3/4 & -2 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} -17/2 \\ -17/2 \end{bmatrix}$$

The solution for p_1 is:

$$p_1 = \frac{\begin{vmatrix} -17/2 & 3/4 \\ -17/2 & -2 \end{vmatrix}}{\begin{vmatrix} -2 & 3/4 \\ 3/4 & -2 \end{vmatrix}} = \frac{23.375}{3.4375} = 6.8$$

which is equal to p_2 .

Bi-Market Monopoly

Next consider what happens if both markets are monopolized by the same economic agent. The bi-market monopolist will maximize profit defined in the following way:

$$\max_{\{q_1, q_2\}} \pi = p_1 q_1 - \int m_1 dq_1 + p_2 q_2 - \int m_2 dq_2$$

The FOC are:

$$\frac{\partial \pi}{\partial q_1} = p_1 + q_1 \frac{\partial p_1}{\partial q_1} + q_2 \frac{\partial p_2}{\partial q_1} - m_1 = 0$$

$$\frac{\partial \pi}{\partial q_2} = p_2 + q_2 \frac{\partial p_2}{\partial q_2} + q_1 \frac{\partial p_1}{\partial q_2} - m_2 = 0$$

In this problem it becomes imperative to invert both demand curves. We do this in the following way:

$$q_1 = 10 - 2p_1 + p_2$$

$$q_2 = 10 - 2p_2 + p_1$$

$$\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} q_1 - 10 \\ q_2 - 10 \end{bmatrix}$$

Solving by Cramer's Rule gives:

$$p_1 = \frac{\begin{vmatrix} (q_1 - 10) & 1 \\ (q_2 - 10) & -2 \end{vmatrix}}{\begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix}} = \frac{-2q_1 + 20 - q_2 + 10}{3} = 10 - \frac{2}{3}q_1 - \frac{1}{3}q_2$$

Similarly, the Marshallian demand curve for market 2 is:

$$p_2 = 10 - \frac{2}{3}q_2 - \frac{1}{3}q_1$$

Substituting into our FOC gives:

$$\frac{\partial \pi}{\partial q_1} = (10 - \frac{2}{3}q_1 - \frac{1}{3}q_2) + (-\frac{2}{3}q_1) + (-\frac{1}{3}q_2) - (2 + q_1) = 0$$

Collecting terms:

$$\frac{\partial \pi}{\partial q_1} = 8 - \frac{2}{3}q_1 - \frac{2}{3}q_2 = 0$$

Repeating for the second FOC gives:

$$\frac{\partial \pi}{\partial q_2} = 8 - \frac{2}{3}q_1 - \frac{2}{3}q_2 = 0$$

In matrix form, we have:

$$\begin{bmatrix} \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 8 \\ 8 \end{bmatrix}$$

Solving by Cramer's Rule gives $q_1 = q_2 = 2.67$. Substituting into the Marshallian demand curves we have price in the two markets equal to 7.3.