ON THE QUESTION OF SEPARABILITY

What we would like to be able to do is estimate demand curves by segmenting consumers' purchases into groups. In one application, we aggregate purchases by consumption category, such as food, housing, etc., instead of looking at each item of food purchased by a consumer. Or in another setting, as in the dog food example, we estimate demand using the budgeted expenditures for a particular class of products as the constraint on behavior. The theoretical consistency of this approach is the question we pursue today.

Consider a utility function

$$U = U(x_1, x_2, x_3, \ldots, x_n) \qquad 1$$

where we want to combine the goods into categories:

$$\begin{aligned} y &= f(x_1, x_2) \\ z &= h(x_3, \ldots, x_n) \end{aligned} \qquad 2$$

Let p_z be the price of z and p_1 and p_2 be the prices of the goods making up y.

In general there is no problem in grouping goods. For instance if $z = \sum_{i=3,n} p_i x_i$, then p_z can be interpreted as a scalar for the p_i such that $p_z z = \sum t p_i x_i = t \sum p_i x_i$. Utility is a function of z, which is really an expenditure value. In this sense we can group all other goods into one composite commodity where we examine the effect of changes the price level of this composite. These changes are proportional adjustments of all component prices. Relative prices are constant. Changes in the price of the single good have an effect on the consumption of the composite which is really the expenditure on all other goods. This is the framework that underlies our analysis of the labor leisure tradeoff and it is perfectly acceptable.

The more troublesome question is posed by the "Problem with Dogs." We must address the question, Are ordinary demand curves using the budget share of y as a constraint, logically consistent with our standard model of consumer behavior? This is the problem of *empirical* separability.[1] In other words, can we effectively ignore z in the construction and estimation of demand curves for the other goods, x_1 and x_2?

The utility maximizing model using a budget group constraint looks like this:

$$\max_{\{x_1, x_2\}} U = (x_1, x_2, \bar{z}) \text{ s.t. } B = \left[M - p_z \bar{z}\right] = p_1 x_1 + p_2 x_2 \qquad 3$$

which gives ordinary demand curves that look like:

$$x_i^{\,B} = x_i^{\,B}(p_1, p_2, B) \qquad 4$$

compared to our normal demand curves:

[1] I am coining the phrase, *empirical separability*, not to be confused with separable utility functions.

$$x_i^M = x_i^M(p_1, p_2, p_z, M) \qquad 5$$

The standard textbook treatment focuses on utility functions and it is not very empirically oriented even though the question is basically a practical empirical issue.[2] That is, when can we effectively ignore all but a subset of consumer behavior and still get demand curves that are logically consistent. Since we do this all the time, it is important to know what assumptions we are implicitly making and what consequences follow if the world deviates from those assumptions.

As a purely empirical matter, if

$$\frac{\partial x_i^M}{\partial p_z} = 0 \qquad 6$$

for all *i*, then we have empirical separability. That is the demand effects estimated from eqt (4) will be identical to those estimated using eqt (5).

It is reasonable to ask, When is it likely that eqt (6) holds?

We know from the budget constraint conditions that:

$$(\varepsilon_{zz} + 1)S_z + \varepsilon_{yz}(1 - S_z) = 0$$

If eqt (6) is true for all *i*, then the cross price elasticity of *y* with respect to the price of *z* is zero. This means that the own-price elasticity of *z* must be –1.

This says that if the set of goods *z* is expenditure neutral with respect to a price index on this set, then the set of goods *y* can be treated as separate. Moreover, if we estimate demand curves for the set *y* and find that at the weighted average expenditure elasticities and own-price elasticities are unitary, then we can logically infer that ε_{yz} is zero.

Empirical separability means that all the consumer behavior relations that we derived for the set of all goods apply to a subset standing alone. This will be true if all cross price effects in the subset with respect to the rest are zero. This will be the case if the own price elasticity of all other goods is unitary, and it will be known to be the case if when we estimate the demand system for the subset, the weighted average own-price and income elasticities are unitary.

A look at the Chinese household demand curves gives us an idea of the application. When I estimated the demand curves for all goods, food was found to have a unitary price and income elasticity. Cross price effects with all other goods were not statistically different from zero. In this setting, if food as a group has no cross price effects, then food is separable. Hence, I could have estimated demand for the individual food groups by looking only at the expenditures on food, and ignoring the expenditures on all other goods.

Probably the most important application of the separability result is recognizing what we cannot do and why. Consider the data that BiLo collects when you make purchases there and use your BiLo discount card. The company knows all of your purchases, item by item, and your total expenditure for each visit. Can they estimate demand curves for you for different items? (Could they target you for particular discounts using only these data?)

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[2] See Layard & Walters pp. 165-167. Silberberg (1990) (2000) section 10.7, section 1, Varian (1992) Chapter 9.3, Intriligator, Bodkin, Hsiao (1996)-sections 7.1-7.7. These texts all talk about the conditions under which the utility function is separable.

The answer is "No." The problem is that of the items that you buy at BiLo, you may and probably do shop around for the best price among several stores. BiLo only sees your purchases when its price on some or all of the items purchased is most attractive. When you don't buy something at BiLo, it does not necessarily mean that you consume zero quantity of that item.

Technically, the BiLo data give us a biased sample in multiple ways. Obviously, consumption is biased, but so is total expenditure. Since expenditures are higher at BiLo when it has favorable prices on select items, purchases of those items will soak up some (all?) of the income effect.[3]

Some derivations:

A. Weighted sum of own and cross price elasticities should equal –1.

It has to be true because if prices increase and income doesn't, total expenditures are constant even though the optimal bundle may change.

Add the budget constraint conditions and collect terms:

$$S_1(\varepsilon_{11}+1)+S_2\varepsilon_{21}+S_2(\varepsilon_{22}+1)+S_1\varepsilon_{21}=0$$

$$S_1\varepsilon_{11}+S_2\varepsilon_{21}+S_2\varepsilon_{22}+S_1\varepsilon_{21}=-S_1-S_2$$

$$S_1(\varepsilon_{11}+\varepsilon_{12})+S_2(\varepsilon_{22}+\varepsilon_{21})=-1$$

B. Estimated budge elasticity from category expenditures is equal to estimated income elasticity when empirical separability holds.

Start with the equality of the unrestricted, restricted, and compensated demand curves:

$$x_i^M(p_1,p_2,p_z,\hat{M})=x_i^B(p_1,p_2,\hat{B})=x_i^U(p_1,p_2,p_z,U)$$

The hat markers on income and the subset budget for *y* signify an income level that satisfies duality.

Differentiate with respect to P_z:

$$\frac{\partial x_1^M}{\partial p_z}+\frac{\partial x_1^M}{\partial M}x_z=\frac{\partial x_1^B}{\partial B}\left(-\frac{\partial x_z^M}{\partial p_z}p_z\right)=\frac{\partial x_1^U}{\partial p_z}$$

[3] In some simulations using Cobb-Douglas preferences, I found the BiLo cross price effect to completely erase the income effect when estimated in unrestricted form. The homogeneity condition was not satisfied. Imposing the homogeneity condition reduced the bias. The estimated income elasticity was .88, own price was –1.13, and cross price was .25. These, of course, should be 1, -1, 0. The estimated budget share was greater than one.

Recall that $\hat{B} = \hat{M} - p_z x_z^M$, which accounts for the middle term; and $\frac{\partial x_1^M}{\partial p_z} = 0$ by the definition of separability.[4]

Hence, we have:

$$\frac{\partial x_1^M}{\partial M} = \frac{\partial x_1^B}{\partial B}\left(-\frac{\partial x_z^M}{\partial p_z}\frac{p_z}{x_z}\right)$$

and because the own-price elasticity of *z* is –1, we have the result. Moreover, if the income elasticity are equal, the price elasticities are as well.

The Labor-Leisure Model

It is worth looking at the labor-leisure model right now because it is an application of the separability question. That is, we commonly look at systems of demand equations without looking at the labor-leisure choice. For instance, in the Chinese demand estimates we held constant total expenditures on goods and not total potential income.

The standard labor-leisure model takes the form:

$$\max_{\{C,l\}} U = U(C,l) + \lambda(E + (T-l)w - C)$$

where *C* is the composite consumption good, *l* is leisure, *T* is total time, *E* is endowment income, and *w* is the wage rate. Labor (unlabeled) is *T - l*. Endowment income can be zero or even negative. However it is both intuitively pleasing and parsimonious to model the budget constraint so that consumption of goods, *C*, is equal to endowment income plus labor income. We will see why shortly. By the way, it can be seen directly from the budget constraint that the wage rate is the opportunity cost of leisure because a change in the dollar value of consumption is equal to the negative of a change in leisure times the wage rate.

The FOC and SSOC of the maximization problem imply demand curves that we are familiar with:

$$C^* = C^*(w,E,T)$$
$$l^* = l^*(w,E,T)$$

Next, let's take a look at the dual. Here we will minimize *E* subject to a utility constraint, i.e.,

$$E = C + lw - wT + \mu(\overline{U} - U(C,l))$$

The FOC are:

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4 $\frac{\partial\left[\hat{M} - p_z x_z\right]}{\partial p_z} = \frac{\partial \hat{M}}{\partial p_z} - x_z - p_z\frac{\partial x_z}{\partial p_z} = x_z - x_z - p_z\frac{\partial x_z}{\partial p_z}$

$$\frac{\partial E}{\partial C} = 1 - \mu U_C = 0$$
$$\frac{\partial E}{\partial l} = w - \mu U_L = 0 \qquad 7$$

which along with the SSOC imply demand curves that take the form:

$$\hat{C} = \hat{C}(w, \overline{U})$$
$$\hat{l} = \hat{l}(w, \overline{U}) \qquad 8$$

Based on the duality theorem we can equate the two sets of demand curves by properly choosing the parameters that demand is a function of. Thus, the duality theorem says:

$$\hat{l}(w, \overline{U}) = l^*(w, \hat{E})$$

Differentiating,

$$\frac{\partial \hat{l}}{\partial w} = \frac{\partial l^*}{\partial w} + \frac{\partial l^*}{\partial E}\frac{\partial \hat{E}}{\partial w}$$

Recognize that the demand curves in (16) imply an optimized expenditure level:

$$\hat{E} = \hat{C}(.) + \hat{l}(.)w - Tw$$

Thus, by the envelope theorem,

$$\frac{\partial \hat{l}}{\partial w} = \frac{\partial l^*}{\partial w} - \frac{\partial l^*}{\partial E}(T - l^*) \qquad 9$$

This result is a little odd. It is the Slutsky equation for the labor-leisure tradeoff. It says that the pure substitution effect for leisure (the left-hand side) is equal to the ordinary price effect *minus* the weighted income effect. The minus sign come in because of the way the budget constraint is specified.

The Slutsky equation in (18) makes sense nonetheless. We know that the pure substitution effect must be negative. This can be derived from the comparative static analysis of the FOC of the endowment minimization equations given in (17). Notice that the wage rate only shows up in one place by itself so the matrix of comparative statics is symmetric.

The right-hand side of (18) is a price effect and an income effect. If the price effect is negative, then leisure is a normal good. As with all normal price effects, the income effect can be positive or negative (just not too big). On the other hand, if the price effect is positive, we have what is known as the "backward bending supply curve of labor." This is the equivalent of a Giffen good in leisure. The more leisure costs, the more the consumer chooses. If the price effect of leisure is positive, equation (18) says that the income effect must also be positive. That is, if the consumer chooses more leisure when the wage rate goes up and it costs more not to work, then the consumer must choose more leisure when endowment income goes up. Indeed that effect (weighted) must be even stronger than the positive price effect.

Having thought about the labor-leisure problem in this way, let's consider the separability question. The issue is essentially the following: Is it legitimate to ignore the labor-leisure tradeoff when estimating demand curves (singly or systems of equations)?

The answer is yes, almost by construction. Consider the model:

$$\max_{\{x_1,x_2,l,\lambda\}} u = U(x_1,x_2,l) + \lambda(P_1x_1 + P_2x_2 - E - (T-l)w)$$

From this we get ordinary and compensated demand curves. The endowment income level that equates them for a given set of parameters is labeled with a hat.

$$x_i^E(P_1,P_2,w,\hat{E}) = x_i^U(P_1,P_2,w,U)$$

Next consider demand curves that come from a model that just looks at expenditures called income in the normal parlance:

$$\max_{\{x_1,x_2,\lambda\}} u = U(x_1,x_2) + \lambda(M - P_1x_1 - P_2x_2)$$

where

$$M = P_1x_1^E + P_2x_2^E$$

and from which we get our standard ordinary demand curves:

$$x_i = x_i^M(P_1,P_2,M)$$

In order for these standard demand curves estimated by ignoring leisure and wage rate to be consistent, it must be true that the derivatives of the standard demand curves be identical with the full model at the point where the ordinary and compensated demand curves intersect. That is, at:

$$x_i^E(P_1,P_2,w,\hat{E}) = x_i^U(P_1,P_2,w,U) = x_i^M(P_1,P_2,\hat{M})$$

the derivates of x_i^E and x_i^M with respect to the P_i must be equal and

$$\frac{\partial x_i^E}{\partial E} = \frac{\partial x_i^M}{\partial M}$$

This is true by construction: $\hat{E} = P_1x_1^E + P_2x_2^E - (T-l^E)w$ and $\hat{M} = \hat{E} + (T-l^E)w$. Hence, differentiating with respect to E yields:

$$\frac{\partial x_i^E}{\partial E} = \frac{\partial x_i^M}{\partial E} = \frac{\partial x_i^M}{\partial M}$$

If the income effects are equal, then the price effects will be also.