Consumer's Surplus[1]

The idea of consumer's surplus is to attempt to measure in money terms the value of consumption of a good from the information contained in the demand curve. The notion is to take the area under the demand curve and attribute to this the welfare implication that this is how much the good is worth to the consumer in total.

The idea seems to make sense when we think about utility constant demand curves of the sort $x_i^U(.)$. Let the price of the ith good go from high, P_1, to low, P_0. The area under the utility constant demand curve is:

$$- \int_{P_0}^{P_1} x_i^U \, dP_i \tag{1}$$

Recall that from the envelope theorem the derivative of the minimized expenditure level (M^*) w.r.t. price in the utility constant problem is the consumption level of the ith good.[2] From this we can write:

$$- \int_{P_0}^{P_1} x_i^U \, dP_i \;=\; - \int_{P_0}^{P_1} \frac{\partial M^*(P,U^0)}{\partial P_i} \, dP_i \tag{2}$$

Hence, the definite integral is well defined as:

$$- \int_{P_0}^{P_1} x_i^U \, dP_i \;=\; - M^* \bigg|_{P_0}^{P_1} = M^*(P_0) - M^*(P_1) \tag{3}$$

The integral to the left of the utility constant demand curve between two prices can be interpreted as the amount of money that a lower price P_0 is worth to the consumer starting from the higher price, P_1. That is, how much money will the low price reduce the consumer's expenditures assuming that the consumer stays on the same indifference curve.

Operationally, however, this is not obviously useful. The problem is that we cannot observe utility and, hence, cannot hold it constant. On the other hand, observable demand curves are based on holding money income constant. Unfortunately these demand curves do not have the property of definite integration found in the utility constant demand curves (see appendix). That is, we cannot identify the area under the money constant demand curve as meaning anything.[3] Of course, if the income effect for good i is zero, then the ordinary demand curve lies on top of the compensated demand curve and the area under one is the same as the other. Even

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[1] See Silberberg (1990) p. 396- 405; p. 596 - 600, section 11.5; Silberg (2000) p 357-367; p. 600-604; Layard and Walters (1978) p. 146-156; Varian (1992) Chapter 10; Nicholson (1998) pp. 152-157; Nicholson (2000) p. 107-109.

[2] Stars are used to designate the functions expressing optimized values in terms of the parameters of the problem. Where there is some ambiguity regarding the arguments in these optimized functions, as is the case in terms of the demand curves, then more specific notation is used. The arguments of these functions are not generally explicitly noted.

[3] The problem from an old comprehensive examination involving corn and rats highlights this point. A formal derivation is found in the appendix to this lecture.

so, we still appeal to the integration of the compensated demand curve, as given in eqt. (3) above, for the interpretation of what this are means.

      While the area under the ordinary demand curve does not mean anything, we are not completely at a loss. From the Slutsky equation, we know that the slope of utility constant demand curves can be computed from the money constant ones. The amount that a consumer will pay for a lower price is computed along the money constant demand curve by determining the shift in the money constant demand curve that results from the lump sum payment to get the lower price.

      Consider Figure 1. Shown there is a price change in good x from p^0 to p^1. The consumer starts at point a and moves to b. Point c is the consumer's choice given a compensating adjustment in income that would keep the consumer on the original utility contour. In this sense we can call that income shift, measured in y units in the vertical plane (because the price of y stays constant), the amount that the consumer would have to be paid to willingly accept the higher price p^1 starting from the price p^0. The vertical distance between c' and b' is this y unit measure. This compensating income adjustment is also measured as the area under the compensated demand curve. The compensated demand curve is pictured in the second frame as the line segment ac. The area "under" it is really the area to the left because we draw demand in a Marshallian way. The ordinary demand curve is given by the line segment ab.

      An alternative measurement is the compensated demand curve shown in the second frame of Figure 1 as bd. This is the compensated demand curve built along the utility contour associated with the consumer's second choice, the choice conditioned on the new price p^1. This compensated demand curve answers the question, how much would the consumer be willing to pay to keep the original price p^0 instead of suffering the higher price p^1? The answer is the area under the compensated demand curve bd between the two prices. This area is also measured in the top frame in y units as the vertical distance between the points a' and d'. Though it is not precisely seen in the top frame, the area under the compensated demand curve bd is less than the area under the compensated demand curve ac. In general, consumers must be compensated more to willingly accept a price increase than they will pay to avoid such a fate. This asymmetry of consumer behavior which is a commonplace observation is not an anomaly but rather a result of consumer theory.[4]

      Notice that the area under the ordinary demand curve, ab in the lower panel, measures nothing in this discussion. However, it is an approximate average of the areas under the two different compensated demand curves. Moreover, the slope of the ordinary demand curve along with its income effect gives us a measure of the slope of the compensated demand curves via the Slutsky equation.

      On this basis, I will define *Consumer Surplus* as the area under the compensated demand curve. This is not a unique definition for this term. Some people define consumer surplus differently. Moreover, many people call the area under the compensated demand curve something else as well. The student is simply alerted to this inconsistently in the jargon of

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[4] For instance, it has often been noted in experimental surveys that the payment consumers require to willingly participate in a drug test that has a 1:100 chance of killing them is larger than the amount that consumers will pay to acquire a drug that will completely cure them when they have a 1:100 chance of dying. Rather than being evidence of irrational behavior, we have shown that this behavior is completely predictable based on standard consumer theory.

economics and is counseled to simply state one's own definition when speaking about this problem.

I have adopted the area under the compensated demand curve as consumer surplus because it is logically consistent and arguably observable from information contained in the ordinary demand curve. Moreover, it has implications about the behavior of individuals.

Consumer Surplus and Price Indices

The question of consumer surplus is closely linked to the notion of price indices. There are two standard definitions of price indexes. One is called the Laspeyres index. It is defined as

$$\frac{\mathbf{p}^1\mathbf{x}^0}{\mathbf{p}^0\mathbf{x}^0},$$

that is, the effect of the price change is defined in terms of the original consumption bundle designated by $\mathbf{x}^0$, where $\mathbf{p}^0$ is the original price vector and $\mathbf{p}^1$ is the new set of prices. The other price index is Paache.[5] Here the price change is defined in terms of the new consumption bundle.

Figure 2 uses our two good graph to identify the Laspeyres price index. Let the price of *x* go up so that the consumption choice shifts from *a* to *b*. The effect of the price change as measured by the Laspeyres price index is shown in the lower panel as p^1-p^0 times x^a. This is the large rectangle extending outside of the demand curves. This income loss is also shown in the upper panel as *t-r*. The mitigating effect of the consumer's reaction in choosing less *x* is *t-s* so that the true loss due to the higher price of *x* is *s-r*. Of course, this is the area under the compensated demand curve *ac* in the lower panel. Notice that calculating the loss by the area under the ordinary demand curve understates the loss in consumer welfare.

Compensating and Equivalent Variations[6]

A consistent definition of the terms compensating and equivalent variations can be found using the area under the compensated demand curve. Let's start with the definitions used by Layard & Walters on p. 151:

> The compensating variation (CV) is the amount of money we can take away from an individual after an economic change, while leaving him as well off as he was before it.

> The equivalent variation (EV) is the amount of money we would need to give an individual if an economic change did not happen, to make him as well off as if it did.

The difference between CV and EV can be thought of in terms of the claim on the state of nature that is ascribed to the consumer. For CV, the consumer has a claim to the prior state of nature; in EV, the latter. For the price change shown in Figure 1, if the initial position is *a*, then CV defines

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[5] Donald McClosky claimed that Laspeyres is pronounced 'la spairce' which rhymes with 'scarce'. Paasche rhymes with squash.

[6] Layard and Walters (1978) p150-154; Varian (1992) section 10.1.

the consumer's welfare change from the higher indifference curve, while EV defines the welfare change in reference to the lower indifference curve.

Note that both CV and EV can be positive or negative, but they are both of the same sign. Their sign is the same as the direction of the welfare change. Again from Figure 1, a price change from *a* to *b* is a welfare decline. Both CV and EV are negative. CV is given by the area under the compensated demand curve shown in the lower panel as the line segment *ac*. It is negative. EV is the area under the compensated demand curve *bd*. CV is a bigger negative number than EV, but this means that in real number space EV>CV.

If the price change were reversed so that prices fell from point *b* to point *a*, then welfare would be increasing. Both CV and EV would be positive. CV would be defined in terms of the lower utility level; EV in terms of the higher. CV would then be the area under the compensated demand curve *bd* while EV would be the area under *ac*. EV>CV.

The rule is: For normal goods and measured in both positive and negative numbers, EV is always bigger than CV.

Consumer Surplus and All-or-Nothing Demand[7]

It is clear that the precise application of consumer surplus to a particular problem depends critically on the precise question that is being asked. To add to this basic confusion, the definition of consumer surplus varies from one writer to the next. The convention adopted here is to define consumer surplus as the area under a compensated demand curve. This is the definition that Silberberg uses as well as Layard & Walters. However, Layard & Walters use the term a little more restrictively.

For them, consumer surplus is the price-quantity rectangle under an *all-or-nothing* demand curve. All-or-nothing demand answers the question, What is the maximum amount that a consumer would pay for a particular quantity of *x*? Or, what is the most that a seller can possibly charge for a particular block or bundle of *x*? Say the consumer would pay R^1 for x^1. This represents a point on the all-or-nothing demand with x^1 on the horizontal axis and $\{R^1/x^1\}$ on the vertical axis. The intercept along the vertical axis is called the *exclusion price* (a concept that we will discuss more in a moment).

Layard & Walters measure the all-or-nothing demand in the following way. I have reproduced the Layard & Walters graph (their figure 5-11, p. 149) in Figure 3; it is not drawn exactly as they do as I will explain in a moment. By their definition, the all-or-nothing payment that the consumer is willing to pay for x^0 units of *x* is given by the distance *ab*. The argument goes like this: If the consumer were allowed to buy *x* at a price associated with the slope of the line *ma*, call this *p*, the consumer would choose x^0 units. The consumer would be on the indifference curve at *m* if priced out of the *x* market. (The exclusion price is a price high enough so that the slope of the budget constraint makes *m* the optimal choice for the consumer.) The vertical distance between the two indifference curves, *ab*, is the *y* measure of the value of this consumption choice. That is, *ab* represents the maximum amount that a consumer would be willing to pay in addition to how much is already being spent in order to have the good, px^0.

Their argument seems to make sense, but careful inspection of Figure 3 shows that it is not true in this picture. In Figure 3 the consumer would be willing to pay *cm* in order to have the

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[7] Layard and Walters (1978) p. 149-150.

right to buy the good at the price p. This is the area under the compensated demand curve starting at the exclusion price down to price, p. Since $cm > ab$, to say that ab is the maximum the consumer would pay for the right to buy x at the price, p, is to say that x is a bad from x^1 to x^0, which it is not. So the argument appears flawed.

It turns out that in their figure, x is depicted as having an income elasticity of zero. Hence, x^1 and x^0 are the same point. Moreover, in their mathematical derivation, they do define the all-of-nothing demand as the area under the compensated demand curve, so everything is ok.

As a point of interest, Layard & Walters define compensating and equivalent variations as the distances mc and em, which are consumer surplus measured as the integrals under the compensated demand curves for the two indifference curves relative to the exclusion prices. Layard & Walters claim that the distance mc is bounded, that is, it cannot be infinitely large, whereas, em can be. Implicitly, they are assuming that point m has been observed and point e has not. Obviously if point e exists, the equivalent variation is bounded. If there is no exclusion price for the indifference curve upon which point a lies, then the equivalent variation is unbounded. The question hinges on the value of the exclusion price: If the indifference curve is asymptotic to the vertical axis, the exclusion price is infinite and the value of both the compensating and equivalent variations are infinitely large.

Summary of Consumer Theory

When you came into Ph.D. price theory, one of the things that you were supposed to know is the principles level, text book treatment of the income and substitution decomposition of a price change. This analysis is portrayed in commodity space. In detail you were supposed to understand all the possible wiggles and waggles of this analysis. Second, you were supposed to know how to deal with demand curves in what might be called practical terms. For instance, if you were given a linear demand curve, you were supposed to be able to derive the marginal revenue function, compute elasticity, and identify the area under the demand curve in terms of marginal revenue and elasticity.

In this class we concern ourselves with the precise mapping between these two sets of analysis. A part of the income-substitution analysis that is only shown in some tests is the quantity mapping of the price and substitution effects to price and consumption space. That is, if you portray a price and a substitution effect in commodity space for the good on the horizontal axis and then draw the related price and consumption points below, you are now supposed to know exactly how these curves are related. They are related through the Slutsky equation.

This relation is not trivial. Its precision is required in, for instance, the computation of consumer's surplus as discussed above.

MTM
10/96

An Appendix on Integrating Ordinary Demand Curves

The area under the money constant demand curve is:

$$- \int_{P_0}^{P_1} x_i^M \ dP_i \qquad\qquad \textbf{4}$$

From the envelope theorem the derivative of the maximized utility level w.r.t. price in the money constant problem is the consumption level of the *i*th good times the optimized value of the lagrangian multiplier in that problem, $\lambda^*(P, M)$. So here we can write:

$$- \int_{P_0}^{P_1} x_i^M \ dP_i \ = \ \int_{P_0}^{P_1} \frac{1}{\lambda^*} \frac{\partial U^*(P, M^0)}{\partial P_i} \ dP_i \qquad\qquad \textbf{5}$$

The integral on the right-hand side of eqt. (5) is well defined only if we can factor l^* out of the integral

$$\int_{P_0}^{P_1} \frac{1}{\lambda^*} \frac{\partial U^*(P, M^0)}{\partial P_i} \ dP_i \ = \ \frac{1}{\lambda^*} \int_{P_0}^{P_1} \frac{\partial U^*(P, M^0)}{\partial P_i} \ dP_i \ = \ \frac{1}{\lambda^*} \left[U^*(P, M^0) \bigg|_{P_0}^{P_1} \right] \qquad \textbf{6}$$

This factoring is only possible if λ^* is not a function of P_i, that is, only if λ^* is a constant. If it is then eqt. (6) says that the integral under the money constant demand curve is the money equivalent of the change in utility received as the price of the good changes.

We have not talked much about λ^*, but we know from the envelope theorem that it is the marginal change in utility when the money constraint changes, that is, λ^* is the *marginal utility of money*. The question, then, is, What does it mean to say that the marginal utility of money is a constant?

In the utility maximization problem, λ^* is a function of prices and money. In fact it is a homogeneous function of degree -1 in these variables. If all prices and money increase by a factor of *t*, λ^* falls by this factor. From Euler's Theorem we can write:

$$\sum_{i=1}^{n} \frac{\partial \lambda^*}{\partial P_i} P_i \ + \ \frac{\partial \lambda^*}{\partial M} M \ = \ -\lambda^* \qquad\qquad \textbf{7}$$

If the marginal utility of money is constant w.r.t. all prices, eqt. (7) says that the elasticity of λ^* w.r.t. *M* is -1.

$$\frac{\partial \lambda^*}{\partial M} \frac{M}{\lambda^*} \ = \ -1 \qquad\qquad \textbf{8}$$

Going back to the partial of maximized utility w.r.t. the *i* th price and w.r.t. *M*, we have

$$\frac{\partial U^*}{\partial P_i} \ = \ -\lambda^* x_i^M \ \text{ and } \ \frac{\partial U^*}{\partial M} \ = \ -\lambda^* \qquad\qquad \textbf{9}$$

The second cross partials are:

$$\frac{\partial^2 U^*}{\partial P_i \partial M} = -\left[\frac{\partial \lambda^*}{\partial M} x_i^M + \frac{\partial x_i^M}{\partial M} \lambda^*\right] = \frac{\partial^2 U^*}{\partial M \partial P_i} = -\frac{\partial \lambda^*}{\partial P_i} = 0 \qquad \textbf{10}$$

which are equal by Young's Theorem. Converting eqt. (10) to elasticity form gives:

$$-\left[\frac{\partial \lambda^*}{\partial M}\frac{M}{\lambda^*} + \frac{\partial x_i^M}{\partial M}\frac{M}{x_i^M}\right] = 0 \qquad \textbf{11}$$

Substituting from eqt. (8) leaves us with:

$$\frac{\partial x_i^M}{\partial M}\frac{M}{x_i^M} = 1 \qquad \textbf{12}$$

This result says that if all income elasticities are equal to 1, then the marginal utility of money is a constant. In other words, in order to integrate money constant demand curves and make any sense out of them, all goods must respond uniformly to an increase in money. This result is true for some homothetic utility functions.[8]

But, even in this case the notion of the area under the money constant demand curve is not particularly meaningful. While it is a consistent dollar measure of utility for some utility functions, it does not answer any real world questions.

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[8] For some, but not all. The standard Cobb-Douglas form utility function is homothetic. However the marginal utility of money is a function of prices. Even though all income elasticities are unitary, the integral of the Marshallian demand curve has no economic interpretation.
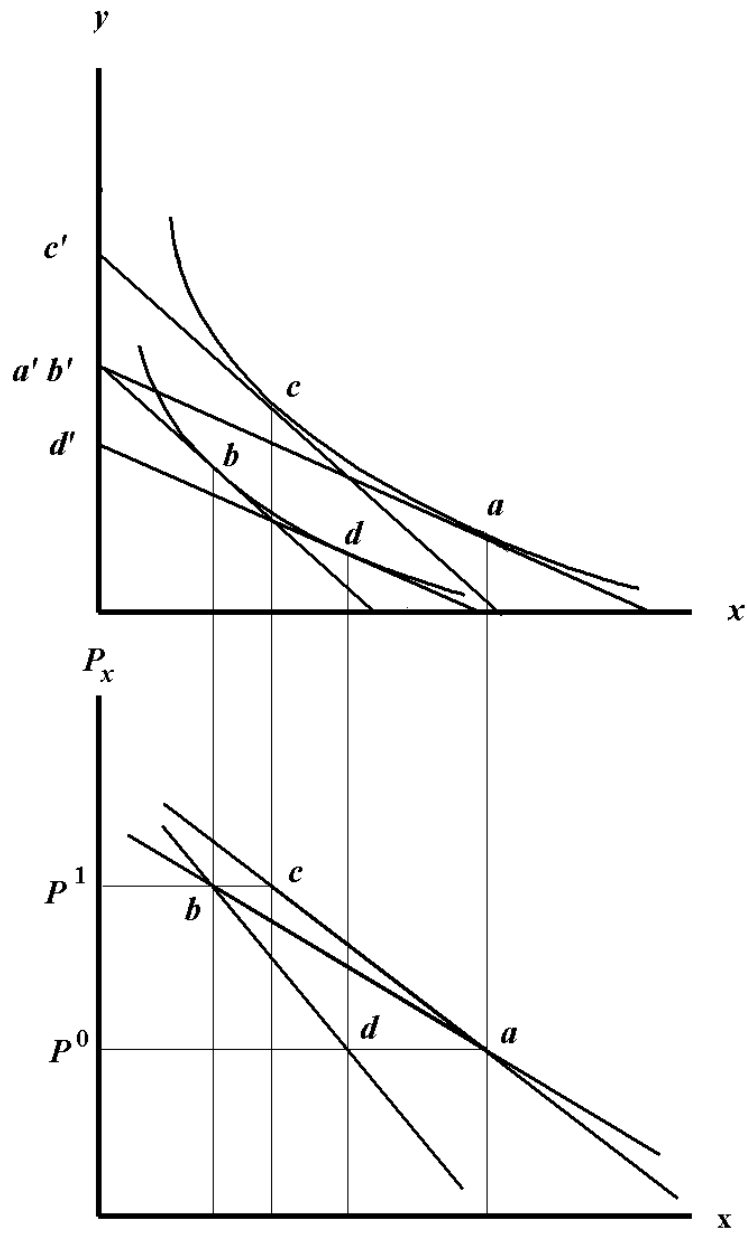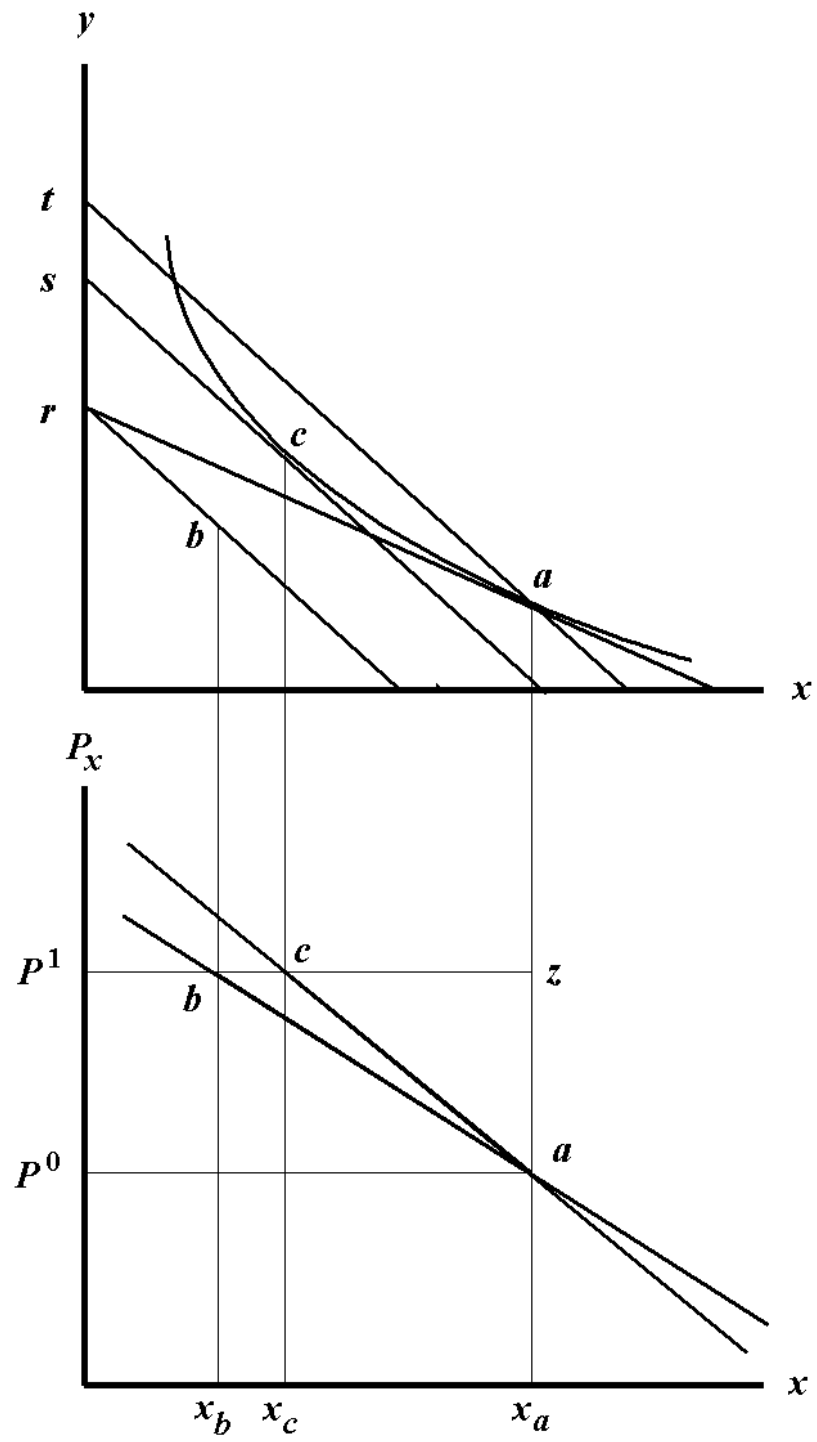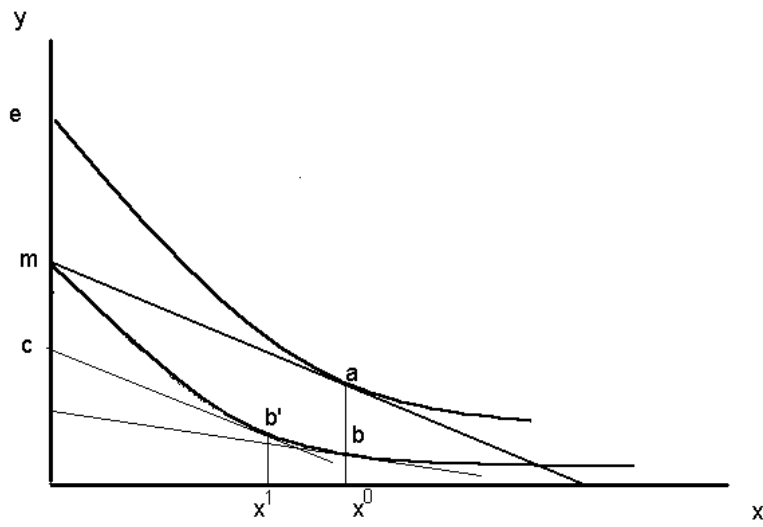
Figure 1

Figure 2

Figure 3

Some notes on the calculation of consumer surplus:

The approach adopted by most people in addressing the problems on p. 17 of the problem list follows the method given by Layard & Walters, bottom p. 146. That is, we calculate the slope or elasticity of the compensated demand curve and then extrapolate for the given price change using a linear approximation.

I want to work through an example and then discuss the alternative of integrating to find the compensated demand curve and consumer surplus.

Consider the case of a price increase of 20%. Let the ordinary own-price elasticity be -1.12, the income elasticity be 1.34, and the consumption share be .13. (This is the example of the price increase in fish under the assumption that fish demand in the United States is the same as meat demand in China.) Assume that income is $31,000. Under these assumptions, compensated own-price elasticity is -.95.

Consumption in dollars before the price change is $4030. A 20 percent price increase would cause expenditures to go up by $806. However, consumers reduce their purchases. Given the compensated price elasticity of -.95, a 20 percent price increase would imply a 19 percent quantity reduction. Hence, the compensated expenditure increase after the price increase would be $653, (i.e., 81 percent of $806). The deadweight loss in consumer surplus, assuming a linear approximation, is ½ (806 – 653), or $76.50. The total compensating variation, or consumer surplus at stake in the proposed price increase is $729.50. [If you draw a simple picture, I think that this will be plainly obvious.]

Now, consider the alternative of integrating to find the compensated demand curve. Since,

$$\varepsilon_{11}^{U} = \frac{\partial x}{\partial P_x}\frac{P_x}{x}$$

we can write

$$\int \frac{\partial x^{U}}{\partial P_x}dp = \int \frac{x}{P_x}\varepsilon_{11}^{U}dp = x^{U}(.)$$

Note that the compensated own-price elasticity is negative, so we must appeal to a constant of integration to get positive consumption at the initial point. Initializing P_x at 1 gives

$$x^{U} = k - A\ln P_x$$

where k is the constant of integration and $-A$ is the product of consumption times the compensated price elasticity. Since price is initialized to 1, the log of initial price is zero, and k is

amount of consumption at the initial point, i.e., 4030. The value of A is $\varepsilon^U{}_{11}$ (=-.95) times 4030 or 3812.

Using this demand curve, we can estimate the quantity reduction due to a price increase by setting P to 1.2. Consumption along the compensated demand curve is 3335, down from 4030 at the initial level. Using a linear approximation between these two points, the loss in consumer surplus due to the increase in price is estimated to be $737.

We can be a little more precise by integrating x^U. The integral of x^U over the proposed price change is:

$$\int_1^{1.2} x^U dp = \int_1^{1.2} (k - A \ln P_x) dp = [kP_x - A(P_x \ln P_x - P_x)]\big|_1^{1.2}$$

where again k is the initial consumption level, 4030, and A is 3812. Evaluating the integral gives $734.

The integration approach gives a slightly higher level of consumer surplus than my basic linear approximation, but all three estimates are close to each other.

MTM
10/14/2003

More on estimating consumer surplus:

Consider the utility function $U = \prod_{i=1,n} x_i$. This gives demand functions of the form:

$$x_i = \frac{1}{n}\frac{M}{P_i}$$

The indirect utility function is

$$U^* = \left(\frac{M}{n}\right)^n \Big/ \prod_i P_i$$

By duality,

$$M^* = nU^{\frac{1}{n}}\left(\prod P_i\right)^{\frac{1}{n}}$$

From which we get

$$\frac{\partial M^*}{\partial P_i} = U^{\frac{1}{n}} \left(\prod_{j \neq i} P_j \right)^{\frac{1}{n}} P_i^{-1+\frac{1}{n}} = x_i^U(.)$$

For simplicity, initialize all prices except *i* to one, and suppress the subscripts. The definite integral of compensated demand is

$$\int_a^b x^U \, dp = U^{\frac{1}{n}} \int_a^b P^{-1+\frac{1}{n}} \, dp = U^{\frac{1}{n}} n P^{\frac{1}{n}} \Big|_a^b$$

Alternatively, we can integrate the Slutsky equation from the ordinary demand. For our demand function the Slutsky equation looks like

$$\frac{\partial x^U}{\partial P} = \frac{\partial x}{\partial P} + x \frac{\partial x}{\partial M} = -\frac{M}{n} \frac{1}{P^2} + x \frac{1}{nP}$$

Substitute the demand expression for *x* in the last term on the right and integrate to get the compensated demand function:

$$\int \frac{\partial x^U}{\partial P} \, dp = \int \frac{1}{P^2} \left(\frac{M}{n^2} - \frac{M}{n} \right) dp$$

which gives an estimate of compensated demand

$$\hat{x}^U = \frac{1}{P} \left(\frac{M}{n} - \frac{M}{n^2} \right) + k$$

where *k* is the constant of integration. The constant of integration is initialized to make x^U equal the observed level of consumption.

The definite integral of our estimate of compensated demand looks like this:

$$\int_a^b \hat{x}^U \, dp = \left[\left(\frac{M}{n} - \frac{M}{n^2} \right) \ln(P) + kP \right]_a^b$$

Let's compare consumer surplus calculated from the exact compensated demand curve with that from the Slutsky estimate. Let *M* be $100,000, *P* be $5,000, *n* be 20, and *x* be 1. (This is something like demand for cars. I suspect that people with $100K income spend around $5K per year on cars.) Now ask how much consumers would have to be compensated to suffer a doubling of the price from $5K to $10K.

From the exact compensated demand curve, the value is $3526. From the Slutsky estimate, the value is $2542. These are not very close because of nonlinearity and the size of the price change. As the price change converges to zero, the two methods give the same answer.

Reconsider the utility function $U = \prod_{i=1,n} x_i$. This gives demand functions of the form:

$$x_i = \frac{1}{n}\frac{M}{P_i} = S_i\frac{M}{P_i}$$

That is, each good takes up $1/n^{\text{th}}$ of the budget.

The indirect utility function is

$$U^* = \left(\frac{M}{n}\right)^n \bigg/ \prod_i P_i$$

and by duality, the expenditure function is:

$$M^* = nU^{\frac{1}{n}}\left(\prod P_i\right)^{\frac{1}{n}}$$

Let the prices of all goods except good i equal one and drop the subscript. Then we can write:

$$CS = M^*\Big|_{P_a}^{P_b} = \frac{(U^*)^S}{S}P^S\Big|_{P_a}^{P_b} = \frac{(U^*)^S}{S}[P_b^S - P_a^S]$$

Substituting for U^*, we have:

$$CS = \frac{\left[(M\cdot S)^{\frac{1}{S}}/P_a\right]^S}{S}[P_b^S - P_a^S] = M\left[\frac{P_b^S}{P_a^S} - 1\right]$$

A problem:

Clemson University offers the following room and board options to students. The average room charge is $2960 per academic year. The most expensive housing is $4230 per year. The least expensive is $2000. The basic meal plan allows students to eat 15 meals per week for $996. (There are a total of 21 meals per week served in the dining halls. The average student eats 15 meals out of 21.) The university estimates that personal expenses amount to $1666 per year.

Assume that ordinary price and income elasticities are (-1) and (+1) respectively for all goods, and that scholarships or parents pay for tuition and educational expenses so that a student's budget is allocated to housing, meals, and personal items.

    a. How much will a student living in the most expensive university housing spend on meals? How much will a student living in the least expensive university housing spend on personal items? (10 points)

    b. The university offers a meal plan that covers all 21 meals per week (*100% meal plan*) for $1072. Is this a good deal for the average student? Explain. (10 points)

    c. What is the maximum price that the university could charge the average student for the 100% meal plan? (15 points)

    d. If parents pay housing costs, so a student's budget is only meals and personal items, what effect does this have on the value of the 100% meal plan? (10 points)

See <ConsumerSurplusCalcs.xls>.