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Queuing Lines and Lists[1]

LINES — BARZEL

1. Why is demand downward sloping?

Given a good with a normal downward sloping market demand where a given quantity supplied is distributed at zero price, a waiting line or queue will form to allocate the supply among the excess demanders. The time cost of being in the queue for the marginal queuer must equal what would be the market equilibrium money price if it were charged. Assume that all demanders have the same opportunity cost of time which is the wage rate, *w*. Hence:

$$t^* = P^*/w$$

Assume that service time is instantaneous. A queue equal to the allotted supply forms t^* hours before service occurs. If wage rates differ across demanders but are known perfectly by all demanders, the queue still forms instantly t^* before service begins and is randomly ordered with people having $w_i\, t^* < P^*$. If there is imperfect knowledge, then the queue will be roughly ordered by w_i. That is the person with the lowest time cost gets in line first and the person with the marginal time cost gets in line last.

2. Why does service time not affect time in queue, at least in general?

If there is a service time, then the last person in the queue arrives exactly at t^*, and there are t^*/t_s -1 people ahead. In this sense there is a difference between the length of the queue and the waiting time in the queue. If there is a service time, the queue forms progressively. However, the time in the queue is still determined by the market clearing money shadow price, which is unaffected by the rate of service. With a service rate, the equilibrium wait is still t^*. A new person enters the queue just as a person leaves the service area.

Faster service in a server queue => longer waiting line, though not a longer wait, i.e., t^*. Remember the comp question on check out lines at the grocery store.

3. Which products are provided by queues?

Inelastic goods. Discussion turns to multiple pass queues and there is an interesting discussion about all-or-nothing demands. Basic result is that a queue is a way of allowing people to work (stand in line) for a good. If the benefit of the queue is to allow these people to sell their time for this good, then area under the demand curve is the measure of total benefits.

Inelastic => heterogeneous in terms of demand interest. That is, some people really like the good while others are vaguely indifferent.

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[1] Barzel, Y. "A Theory of Rationing by Waiting" in Journal of Law and Economics. Vol 17. 1974, p73-93. Layard and Walters (1978) section 6.4.

4. Will some queues be populated by high income people?

Let demand be given by

$$q = \alpha P^{\beta} I^{\gamma}$$

Rewrite demand in Marshallian form

$$P = \alpha^{-\frac{1}{\beta}} q^{\frac{1}{\beta}} I^{-\frac{\gamma}{\beta}}$$

Let $P = wt$ and $w = kI$, then

$$kIt = \alpha^{-\frac{1}{\beta}} q^{\frac{1}{\beta}} I^{-\frac{\gamma}{\beta}}$$

or

$$I = \left[t^{-1} k^{-1} \alpha^{-\frac{1}{\beta}} q^{\frac{1}{\beta}} \right]^{\frac{1}{1+\frac{\gamma}{\beta}}}$$

Assume that q equals 1, that is, that everyone only gets one unit. Differentiating w.r.t. t gives

$$\frac{\partial I}{\partial t} = -\left(\frac{1}{1+\frac{\gamma}{\beta}} \right) \frac{I}{t}$$

This derivative is positive if the income elasticity of demand is greater than the absolute value of price elasticity.

To say that the derivative is positive is to say that as the time price that rations demand goes up, the income level associated with the demand of the marginal queuer goes up. Thus, if the income elasticity of demand is greater than the price elasticity, the queue is populated with high income individuals.

5. How does trade affect the queue?

LISTS — LINDSAY & FEIGENBAUM[2]

1. Value is a declining function of time in the queue. $V_0 e^{-gt} - c$, where V_o is the no cost, no wait value of the good, g is the decay + discount rate, t is the expected wait, and c is the cost to join the queue.

2. People join the queue so long as they receive positive value given the expected wait. (Note c must be positive given the functional form of the decay, but the model could be more generally constructed so that c could be the discounted value of the future cost of the good. Such a formulation will work so long as the cost discount factor is less than the discount plus decay factor on the benefits.)

3. People are distributed on the basis of V_o.

4. For any given t, say t', the people for whom $V_0 e^{-gt'} - c$ is greater than or equal to 0 join the queue. Joining can be described by assuming that the population, P, is normally distributed across V where

$$V(t) = V_0 e^{-gt} - c\,,$$

has a standard deviation of σ and a mean of

$$\mu_{V_0} e^{-gt} - c\,.$$

(μ_{Vo} is the mean of V_o.) Joiners are the cumulative distribution of the normal density $n(V)$ above zero as shown in eqt. (1)

$$j(t) = P\int_{V_0 e^{-gt} - c = 0}^{\infty} n(V; \mu_{V_0} e^{-gt} - c, \sigma)\, dV \qquad \mathbf{1}$$

It is helpful to rewrite this expression in terms of the unit normal density. Let $z = (V - \mu_V)/\sigma$. For $V = 0$, $z = -\mu_V/\sigma$ and eqt. (1) can be rewritten as:

$$j(t) = P\int_{-\frac{\mu_{V_0} e^{-gt} - c}{\sigma}}^{\infty} n(z; 0, 1)\, dz \qquad \mathbf{2}$$

By converting the normal density in (1) into the unit normal, we are able to isolate all of the parameters of the model in the limit of the integral. This makes differentiation almost trivial.

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[2] Lindsay, CM and Feigenbaum, B. "Rationing by Waiting Lists" in American Economic Review. Vol 73. 1984. p404-417.

5. Eqt. (2) is an expression describing the number of joiners per instant in time as a function of the expected time, *t*, spent in the queue; i.e., $j = j(t)$. Differentiating (2) w.r.t. *t* gives:

$$\frac{dj}{dt} = -Pn(z = -\mu_V / \sigma; 0,1)\left(-\frac{1}{\sigma}\right)\frac{d\mu_V}{dt}, \text{ where } \frac{d\mu_V}{dt} = -g\mu_{V_0} e^{-gt} \quad \textbf{3}$$

Clearly, $dj/dt<0$ as shown in eqt. 3. The joiners function, which is like a demand curve for the waiting list, is downward sloping.

6. Let the number of people served at each instant be $s(t)$. L&F assume that $ds/dt>0$. However, it is just as meaningful to assume that this supply curve is perfectly inelastic in terms of *t* as in the Barzel analysis.

7. The intersection of $j(t)$ and $s(t)$ is an equilibrium in the following sense: $j(t)$ gives the addition to queue length at each instant; $s(t)$ gives the reduction; at $j(t^*)=s(t^*)$ the queue length is stable and the equilibrium expected wait is t^*.

8. Comparative Statics: Differentiate the equilibrium condition with respect to each parameter.

a. V_0 falls for all => $j(t)$ shifts down and t^* falls.
b. *g* increases => $j(t)$ pivots inward on horizontal intercept and t^* falls.
c. $s(t)$ increases => t^* falls.

9. Queue length is given by $j(t)t^* = s(t)t^* = Q$. $dQ/dt^* = t^*\, dj/dt + j(t)$. (This is a standard looking inverse marginal revenue function.) Rewrite in elasticity terms: $dQ/dt^* = j(t)[1 + \varepsilon_{jt}]$, where ε_{jt} is the elasticity of j(t) with respect to *t*. This elasticity is negative because dj/dt is negative.

If $\varepsilon_{jt} < -1$, that is, if joining is time elastic, then a reduction in the equilibrium expected wait will cause queue length to increase, e.g., an increase in the service rate can lengthen the queue.

MTM

Adendum

The following graph captures the idea of the waiting list paper. There is a population density representing people demanding the service in each period. This density maps them across the valuation of the service if it were performed immediately. Some people place a high value on the service; some people place a low value. Two people have been highlighted, person *a*, who places relatively low value on the procedure, and person *b*, who has a relative high value.

Everyones' valuation decays as the service is postponed. This is show by the declining curves which decay at $-gt$ and start from the initial valuation points, V_a and V_b.

If the wait is too long, people will not join the waiting list. So, for example, if the wait is greater than t_0, then person *a* will not join the list. This is try because after that time, the value to *a* is less than the cost of signing up, *c*. However, at t_0, person *b* will sign up. Person *b* becomes marginal if the waiting time increases to t_1.

If the waiting time is t_0, then person *a* signs up as well as everyone with initial values greater than or equal to his. This would include all of the people in the blue and red areas of the density. Similarly, if the waiting time is t_1, then everyone with initial valuations greater than or equal to person *b* will sign up. This is shown by the red area of the density.