Cobb-Douglas Cost Functions

The problem is to define cost functions from the Cobb-Douglas production function. We begin by minimizing resource expenditures subject to an output constraint. Output is defined in terms of the inputs according to the Cobb-Douglas function:

$$\text{Min}_{\{K,L\}}\ C = w_K K + w_L L \ \ s.t.\ q_0 = K^{\alpha} L^{\beta} \qquad \mathbf{1}$$

This gives the following first order conditions[1]:

$$\frac{\partial C}{\partial K} = w_K - \frac{\mu\alpha q}{K} = 0 \qquad \mathbf{2}$$

$$\frac{\partial C}{\partial L} = w_L - \frac{\mu\beta q}{L} = 0 \qquad \mathbf{3}$$

$$\frac{\partial C}{\partial \mu} = q - K^{\alpha} L^{\beta} = 0 \qquad \mathbf{4}$$

Solve the first two first order conditions for *K* and *L*.

$$K = \frac{\mu\alpha q}{w_K} \quad L = \frac{\mu\beta q}{w_L} \qquad \mathbf{5}$$

Substitute these into the third first order condition and solve for marginal cost.

$$q = \left(\frac{\mu^*\alpha q}{w_K}\right)^{\alpha}\left(\frac{\mu^*\beta q}{w_L}\right)^{\beta} \qquad \mathbf{6}$$

The star goes on the Lagrangian multiplier now because we have solved all of the FOC simultaneously to get its value. More simplification is enlightening:

$$\mu^* = \left[q^{\alpha+\beta-1}\alpha^{\alpha}\beta^{\beta}\frac{1}{w_K^{\alpha}w_L^{\beta}}\right]^{-\frac{1}{\alpha+\beta}} \qquad \mathbf{7}$$

$$\mu^* = q^{\frac{1}{\alpha+\beta}-1}\alpha^{\frac{-\alpha}{\alpha+\beta}}\beta^{\frac{-\beta}{\alpha+\beta}}w_K^{\frac{\alpha}{\alpha+\beta}}w_L^{\frac{\beta}{\alpha+\beta}} \qquad \mathbf{8}$$

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[1] SSOC hold as the astute student can readily determine.

*K** can be written as:

$$\begin{aligned} K^* &= \frac{\mu^* q\alpha}{w_K} \\ &= q^{\frac{1}{\alpha+\beta}} \alpha^{1-\frac{\alpha}{\alpha+\beta}} \beta^{\frac{-\beta}{\alpha+\beta}} w_K^{\frac{\alpha}{\alpha+\beta}-1} w_L^{\frac{\beta}{\alpha+\beta}} \\ &= q^{\frac{1}{\alpha+\beta}} \alpha^{\frac{\beta}{\alpha+\beta}} \beta^{\frac{-\beta}{\alpha+\beta}} w_K^{\frac{-\beta}{\alpha+\beta}} w_L^{\frac{\beta}{\alpha+\beta}} \\ &= q^{\frac{1}{\alpha+\beta}} \left[\frac{\alpha}{\beta} \frac{w_L}{w_K} \right]^{\frac{\beta}{\alpha+\beta}} \end{aligned} \qquad \mathbf{9}$$

In similar fashion *L** can be solved for.

To find the expression for optimized total cost, *K** and *L** are substituted into the cost identity. This gives:

$$\begin{aligned} C^* &= w_K K^* + w_L L^* \\ &= \mu^* q \cdot (\alpha+\beta) \\ &= q^{\frac{1}{\alpha+\beta}} \alpha^{\frac{-\alpha}{\alpha+\beta}} \beta^{\frac{-\beta}{\alpha+\beta}} w_K^{\frac{\alpha}{\alpha+\beta}} w_L^{\frac{\beta}{\alpha+\beta}} \ (\alpha+\beta) \end{aligned} \qquad \mathbf{10}$$

Price elasticities and cost shares can be derived directly. The own-price elasticity of capital is:

$$\varepsilon_{KK} = -\frac{\beta}{\alpha+\beta} \qquad \mathbf{11}$$

This comes from equation (9) by taking the log and then the derivative w.r.t. the price of capital. The cross price elasticity is the same magnitude as the own price but of opposite sign. Notice that the cross price elasticities are not equal. That is e_{KL} is not equal to e_{LK}. However,

$$\varepsilon_{KL} \frac{w_K K}{C} = \varepsilon_{LK} \frac{w_L L}{C}$$

because the cross price effects are equal.

Similarly, capital's share of total cost can be found by multiplying equation (9) by the price of capital and then dividing by equation (10):

$$\frac{w_K K^*}{C^*} = \frac{w_K \frac{\mu^* q\alpha}{w_k}}{\mu^* q \cdot (\alpha+\beta)} = \frac{\alpha}{\alpha+\beta} \qquad \mathbf{12}$$

Some Notes:

The Lagrangian multiplier is marginal cost. Simply enough this can be verified by taking the derivative of (10) with respect to q; it should be and is identical to the expression given in (9). In a lagrangian problem, the multiplier represents the change in the objective function with respect to the constraint. In this problem the objective is Cost and the constraint is output (q_o). Hence we have marginal cost.

In terms of notation, the stars (*) on μ, K, and C in eqts. (6) through (10) signify that these are optimized values of these variables. They are the values these variables take when (1) is solved given the parameters in that expression. The parameters of the problem are w_K, w_L, α, β, and q_o. You may have noticed that I dropped the subscript from q after eqt. (1). This was purely for notational simplicity—q is still a parameter in the problem defined by equation (1). However, we do not use naught subscripts on all of the parameters and on some only occasionally to emphasize that they are constraints at some phase of the modeling. K and L are the choice variables, and μ* and C* are implied by the optimal choice of K and L. This optimal choice is defined by the solution of (2), (3), and (4) simultaneously. Hence, K in (5) does not have a star, while it does in (9).

The derivative of C* w.r.t. q is μ* and it is called marginal cost.[2] The derivative of μ* w.r.t. q is the slope of marginal cost. Average cost is C* divided by q. The ratio of *AC* to *MC* is (α+β). If (α+β) is one, average and marginal cost are equal. If it is greater than one marginal cost lies below average, and vice versa. Derivatives of C*, *MC*, and *AC* can be taken with respect to all the parameters of the problem. The derivatives of C* and *AC* w.r.t. the input prices are obvious. The derivative of MC with respect to input prices is interesting. All symmetry conditions derived in the general form of the problem hold. The derivatives of C*, *MC*, and *AC* w.r.t. α+β are interesting but really messy. You should solve for all of these.

C*, *MC*, and *AC* are called cost functions. That is, these expressions tell us the optimized value of cost for a target output level and given set of input prices. Having obtained cost functions by means of the foregoing analysis, they are at our disposal to use in other problems, for instance, the problem of profit maximization. In that formulation, output (q) becomes a variable of choice.

Some More Notes:

The derivative of the input demand curve w.r.t. output,

$$\frac{\partial K^*(w_K, w_L, q)}{\partial q}$$

which can be taken from eqt. 9, is not simply the inverse of the marginal product (i.e., the derivative of production function w.r.t. the input). Consider eqt. 9 in logs:

[2] When we take the derivative of solution values of an optimization problem with respect to the parameters of the problem, it is called comparative static analysis. Comparative static analysis is the bedrock of mathematical economics. It is an approach to modeling problems. This mathematical paradigm was developed by Paul Samuelson in his dissertation (*Foundations of Economic Analysis*). He essentially rewrote the corpus of economic theory existing in 1947 casting it in this framework.

$$\frac{\partial \ln K^*}{\partial \ln q} = \frac{1}{\alpha + \beta} = \varepsilon_{K,q} \qquad \textbf{13}$$

This is the elasticity of the input, *K*, w.r.t. output. It is equal to one over the sum of the production coefficients, i.e., one over the degree of homogeneity. On the other hand, the output elasticity of the input is equal to the production function coefficient:

$$\frac{\partial \ln q}{\partial \ln K} = \alpha \qquad \textbf{14}$$

Eqt. 13 answers the question, "How much more capital does the firm use when it attempts to expand output in the most cost effective fashion?" Eqt. 14 answers the question, "How much more capital does it take to expand output holding labor constant. In eqt. 14, labor is held constant. In eqt. 13, labor is varying in the optimal fashion, as is capital as output changes. In this sense, it is important to recognize the there is a big difference between the output elasticity of an input and the input elasticity of output. Indeed, I am not sure that the profession is settled on what these phrases mean. It is certain, however, that when we say "the elasticity of input demand w.r.t. output" we are talking about eqt. 13.

For homothetic production functions, the first order conditions referring to the inputs alone can be solved for some meaningful expressions. This is because the nature of homotheticity means that the optimal input ratio does not change as output is expanded, holding the input demand functions constant. In this sense, the ratio of the inputs given by solving eqt's 2 and 3 is the same as the ratio of the input demand curves as expressed in eqt. 9 for capital and its sister for labor (not shown). Either way, everything cancels and we are left with

$$\frac{K^*}{L^*} = \frac{\alpha}{\beta} \frac{w_L}{w_K}$$

Rewriting this in logs is a way to simply identify the elasticity of substitution. The derivative of the log of the capital/labor ratio with respect to the ratio of capital's price to labor's price is -1, and this is independent of the degree of homogeneity.

Competitive Equilibrium

Cobb-Douglas cost functions of the sort developed thus far are not particularly appealing. If the returns to scale are increasing, the function implies ever falling marginal and average cost, which leads to something akin to natural monopoly. On the other hand, if there are decreasing returns to scale in labor and capital, the marginal and average cost function emanate from the origin. This implies production is free if undertaken in infinitely small applications. Finally, if there are constant returns to scale, production can be accomplished in any size at the same cost.

One way of rescuing the analysis is to assume that cost is composed of variable and fixed components, and that returns to scale are decreasing in the variable components. That is, rewrite (1) so that a fixed cost, *F*, is included:

$$\text{Min}_{\{K,L\}}\ C = w_K K + w_L L + F \ \text{ s.t. } \ q = K^{\alpha} L^{\beta};\ \alpha + \beta < 1 \qquad \textbf{15}$$

Notice that the inclusion of a fixed cost does not affect the FOC, SSOC or any of the primary comparative static results. Total cost can now be broken into a fixed and variable component. The variable component is the same as the cost function derived in (10) above. Average cost is now the sum of average variable cost and average fixed cost. Moreover, average cost is U-shaped. The rising average variable cost[3] is offset by the falling average fixed cost. The U-shaped average cost function is cut at its minimum by marginal cost.

Given a U-shaped average cost function, we can define a competitive equilibrium in terms of market price equal to the minimum of this function. Furthermore, we can write:

$$MC(q^*) = AC(q^*) \qquad \textbf{16}$$

where q^* is the quantity associated with minimum average cost.

This expression can be differentiated w.r.t. one of the variable input prices, say, w_L.

$$\frac{\partial MC}{\partial w_L} + \frac{\partial MC}{\partial q}\frac{\partial q^*}{\partial w_L} = \frac{\partial AC}{\partial w_L} + \frac{\partial AC}{\partial q}\frac{\partial q^*}{\partial w_L} \qquad \textbf{17}$$

However, since average cost is at a minimum by definition, the last term falls out. This leaves the expression we derived in the general case (see eqt 3 in lecture 16), which can be rewritten as:

$$\frac{\partial q^*}{\partial w_L} = \frac{\dfrac{\partial AC}{\partial w_L} - \dfrac{\partial MC}{\partial w_L}}{\dfrac{\partial MC}{\partial q}} \qquad \textbf{18}$$

This familiar expression can be simplified in this case because we know the form of the functions. The easiest way to deal with the C-D cost functions is in log form. Hence, let's convert (18) to percentage changes and then to logs. First multiply both sides by the price of labor; then divide both sides by *q*; finally, multiply the right side by μ^* / μ^* and the first term by AC / AC. These are shown below in brackets.

$$\frac{\partial q^*}{\partial w_L}\left\{\frac{w_L}{q^*}\right\} = \frac{\left\{\dfrac{w_L}{\mu^*}\dfrac{AC}{AC}\right\}\dfrac{\partial AC}{\partial w_L} - \left\{\dfrac{w_L}{\mu^*}\right\}\dfrac{\partial MC}{\partial w_L}}{\left\{\dfrac{q}{\mu^*}\right\}\dfrac{\partial MC}{\partial q}} \qquad \textbf{19}$$

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[3] Where $\alpha+\beta<1$.

Converting the percentage changes to logs gives:

$$\frac{\partial q^*}{\partial w_L}\frac{w_L}{q^*}=\frac{\frac{AC}{\mu^*}\frac{\partial \ln AC}{\partial \ln w_L}-\frac{\partial \ln MC}{\partial \ln w_L}}{\frac{\partial \ln MC}{\partial \ln q}} \quad \mathbf{20}$$

Substituting from the cost functions we have:

$$\frac{\partial q^*}{\partial w_L}\frac{w_L}{q^*}=\frac{(\alpha+\beta)\frac{\beta}{\alpha+\beta}-\frac{\beta}{\alpha+\beta}}{\frac{1-\alpha-\beta}{\alpha+\beta}} \quad \mathbf{21}$$

This simplifies to

$$\frac{\partial q^*}{\partial w_L}\frac{w_L}{q}=-\beta \quad \mathbf{22}$$

This says that the elasticity of the change in competitive equilibrium output w.r.t. the input price is -β. The negative sign of this derivative means that minimum average cost shifts to the left as the input price increases.

Some insight can be gained by returning to the derivative of average cost w.r.t. output:

$$\frac{\partial AC}{\partial q}=\frac{\partial \mu^*}{\partial q}\left[\alpha+\beta\right]-\frac{F}{q^2} \quad \mathbf{23}$$

Substituting for the slope of marginal cost gives:

$$\frac{\partial AC}{\partial q}=\left[1-\alpha-\beta\right]\frac{\mu^*}{q}-\frac{F}{q^2} \quad \mathbf{24}$$

Evaluating this derivative at zero defines the value of marginal cost at minimum average cost:

$$\mu^*\big|_{\min AC}=\mu^*(q^*)=\frac{F}{q^*}\frac{1}{(1-\alpha-\beta)} \quad \mathbf{25}$$

Pay close attention to the notation: marginal cost, μ^*, is a function of q, but marginal cost at minimum average cost is the function μ^* evaluated at q^*.

The value of marginal cost at minimum average cost can be differentiated w.r.t. the wage rate. Substituting from (22) we can solve for the change in the value of minimum average cost:

$$\frac{\partial\left[\mu^*\big|_{\min AC}\right]}{\partial w_L} = \frac{F}{q^*}\frac{1}{(1-\alpha-\beta)}\frac{\beta}{w_L} = \frac{\mu^*(q^*)}{w_L}\beta \qquad \textbf{26}$$

Note that the elasticity of the change in the value of minimum average cost is β, again shown by cross multiplying terms.

Finally, let's consider the market level of output. Market output, call it *Q*, can be described by the market demand function *D*(*P*). The competitive equilibrium price is *P** which is equal to the value of minimum average cost. That is, we assume that the competitive entry/exit process will continue until zero profits prevail. Thus, $Q^*=D(\mu^*|_{\min AC})$ or $Q^*=D(\mu^*(q^*))$. The number of firms necessary to achieve this result is given by the ratio of *Q** to *q**. Call it *n**.

A reasonable question is, What happens to the number of firms as the competitive equilibrium changes because of a change in an input price. We have just demonstrated that, given Cobb-Douglas cost functions with decreasing returns to scale and some fixed cost component, the equilibrium firm size declines as an input price increases. So, too, does market output because the value of minimum average cost increases. If the decrease in the firm size is greater than the decrease in the market quantity demanded, then the number of firms increases. What condition will produce this result?

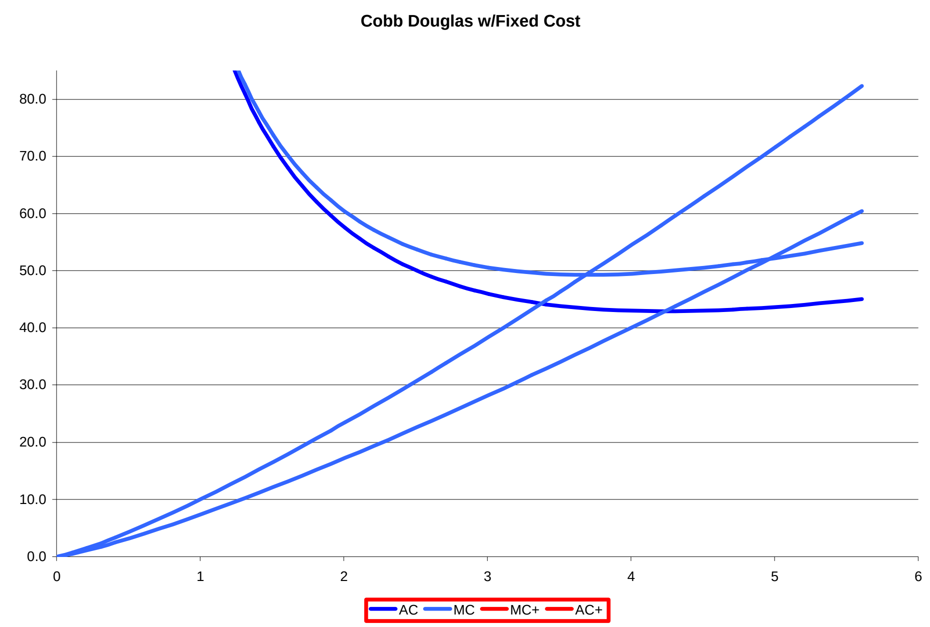

Figure 1

Parameters for Figure 1:

Technical Productivity Coefficient for K	0.2	0.2
Technical Productivity Coefficient for L	0.25	0.25
Price of K	1	2
Price of L	2.5	2.5
Fixed Cost	100	100