■ MULTIVARIATE OPTIMIZING BEHAVIOR[1]

Economics is interesting (and difficult) because we consider problems of choice between two or more things and equilibria between two or more forces. Analytically this means that we need to be facile with multivariate calculus. In order to outline the skills necessary to master our studies let's examine the case of simple profit maximization in the setting of the choice of the mix of two inputs.

• The profit maximizing in the two input case looks like this:

$$\pi = pq - w_1x_1 - w_2x_2,$$
$$\text{where } q = f(x_1, x_2)$$

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where the prices of output, P, and the inputs, w_i, are fixed. Profits are maximized in an unconstrained fashion by the choice of the optimal levels of the two inputs x_1 and x_2. The FOC are:

$$\frac{\partial \pi}{\partial x_1} = \pi_1 = Pf_1(x_1, x_2) - w_1 = 0$$

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$$\frac{\partial \pi}{\partial x_2} = \pi_2 = Pf_2(x_1, x_2) - w_2 = 0$$

3

where $f_i = \partial f / \P x_i$.

The SSOC are based on the hessian determinant composed of the second partials of profits w.r.t. the two inputs:

$$H = \begin{vmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{vmatrix} = \begin{vmatrix} Pf_{11} & Pf_{12} \\ Pf_{21} & Pf_{22} \end{vmatrix}$$

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The requirements for a maximum are that the principal minors alternate in sign starting with negative. In the two-variable case, this means that the main diagonal elements must be negative and the whole determinant positive.

The implications of the model are that the profit maximizing firm operates where price is equal to the ratio of input price to marginal product and where the ratios of input price to marginal product are equal for all inputs. Later on we will show that the ratio of input price to marginal product is marginal cost. However, in the simple profit maximizing model described by eqt. (1), there are no cost functions.

Another way of describing the FOC given by (2) and (3) is to say that the firm operates where its inputs are paid the value of their marginal products.

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[1] Silberberg (1990)-section 4.4; section 6.1-6.2; Layard and Walters Ch 7 pg. 207-213

Comp Question—September 11, 1998, Section I, # 3:

In an industry whose firms produce under conditions of increasing returns to scale, it is not possible to pay each factor of production the value of its marginal product.

      The economics of the problem is this: To say that firms in an industry produce under conditions of increasing returns to scale is to say that the industry will be characterized by a peculiar equilibrium. Increasing returns mean that average cost is falling for each firm. Traditionally such an industry would be labeled "natural monopoly." We expect that one firm would dominate all others if regulatory authorities do not intervene in the process.

      If a monopoly market develops, then the question of paying inputs the value of their marginal product is moot. Monopolists pay the marginal revenue product. Even so, the problem is not without merit. Government intervention in the form of price regulation attempts to stop natural monopolies from extracting consumer surplus. The problem faced by regulators is precisely that it is impossible to force the natural monopolist to pay inputs the value of their marginal products. This problem is neatly characterized in the familiar picture of market demand cutting the average cost curve in its falling range. Because demand cuts marginal cost where it lies below average cost, the regulator cannot force the firm to produce at a price equal to marginal cost without forcing the firm to take losses.

      The model of cost minimization tells us that marginal cost is equal to the ratio of input price divided by marginal product, which is equalized across all inputs. When price is equal to marginal cost, as it is in the case of a competitive equilibrium, then by cross multiplying we have the result that input price is equal to price times marginal product or the value of the marginal product for each input. Hence, if regulators try to force the natural monopolist to produce at the point where demand intersects marginal cost, which would imply paying inputs a wage equal to the value of their marginal product, the sum of these payments, which is average cost times quantity is greater than the revenue, price times quantity.

      The answer to this question can also be demonstrated using Euler's Theorem and assuming a homogeneous production function with increasing returns. Euler's Theorem says that the sum of the first partials of a homogeneous function times the levels of the variables is equal to the degree of homogeneity times the value of the function. That is, where output, q, is a homogeneous function of degree r of two inputs, x_i:

$$x_1 f_1 + x_2 f_2 = rq$$

If the inputs are paid a wage equal to the value of their marginal product, we have:

$$x_1 w_1 + x_2 w_2 = x_1 \cdot pf_1 + x_2 \cdot pf_2 = r \cdot pq$$

If the degree of homogeneity is one, i.e., $r=1$, then input payments exactly equal revenue, pq. However, when the production function is characterized by increasing returns, $r>1$, Euler's Theorem says that if inputs are paid the value of their marginal product, total input payments will exceed revenues.

QED: The answer is true.

■ PROFIT MAXIMIZATION WHEN THE PRODUCTION FUNCTION IS WELL SPECIFIED[2]

It is useful to employ an explicit functional form in this analysis in order to explore some of the dimensions of the first and sufficient second order conditions. To that end, let's substitute the Cobb-Douglas form $x_1^{\alpha} x_2^{\beta}$ for the implicit production function, $f(x_1, x_2)$. In order to develop the model we need only substitute the first and second derivatives of the specific function for the general one.

$$\frac{\partial f}{\partial x_1} = f_1 = \alpha x_1^{\alpha-1} x_2^{\beta}$$

$$f_2 = \beta x_1^{\alpha} x_2^{\beta-1}$$

$$\frac{\partial^2 f}{\partial x_1^2} = f_{11} = \alpha(\alpha-1) x_1^{\alpha-2} x_2^{\beta}$$

$$\frac{\partial^2 f}{\partial x_1 \partial x_2} = f_{12} = \alpha\beta x_1^{\alpha-1} x_2^{\beta-1}$$

$$f_{21} = \alpha\beta x_1^{\alpha-1} x_2^{\beta-1}$$

$$f_{22} = \beta(\beta-1) x_1^{\alpha} x_2^{\beta-2}$$

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The first thing to note is that the second, cross partials of the function are identical. This, it turns out, is not a special case of the function we have chosen, but is a general theorem, called Young's Theorem: cross partials are always identical. This means that we can always write $f_{12} = f_{21}$.

Using the expressions shown in (5), let's now evaluate the FOC and SSOC. The FOC reduce to:

$$\frac{x_2^{*}}{x_1^{*}} = \frac{\beta}{\alpha} \frac{w_1}{w_2}$$

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This says that the ratio of the optimal input levels is a constant based on the production function parameters and the input prices. In other words at any level of output, the optimal capital-labor ratio is fixed. This result is **not** general, but rather follows from the fact that the specific production function we have chosen, the Cobb-Douglas, is a *homogeneous* function.

When we are dealing with explicit functions, the FOC can be solved simultaneously in explicit form. Using the C-D function, we can write:

$$x_1 = \left[\frac{w_1}{P\alpha}\right]^{\frac{1}{\alpha-1}} x_2^{-\frac{\beta}{\alpha-1}} \text{ and } x_2 = \left[\frac{w_2}{P\beta}\right]^{\frac{1}{\beta-1}} x_1^{-\frac{\alpha}{\beta-1}}$$

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Substituting for x_2 in the functional representation of x_1 gives:

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[2] Nicholson (1998)-overview of profit max in Ch 13; varian Ch 13 sections 13.1-13.4

$$x_1{}^* = \left[\frac{w_1}{P\alpha}\right]^{\frac{1}{\alpha-1}} \left[\left[\frac{w_2}{P\beta}\right]^{\frac{1}{\beta-1}} x_1^{-\frac{\alpha}{\beta-1}}\right]^{-\frac{\beta}{\alpha-1}}$$

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I insert the star notation in eqt. (8) to reflect the fact that the value of x_1 is now a functional value based on the simultaneous solution of the FOC. Rewriting and collecting terms, we have:

$$x_1{}^* = \left[\frac{w_1}{P\alpha}\right]^{\frac{\beta-1}{1-\alpha-\beta}} \left[\frac{w_2}{P\beta}\right]^{-\frac{\beta}{1-\alpha-\beta}}$$

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Note that the functional representation of $x_1{}^*$ has as its arguments the economic and technical parameters of the problem.

Next consider the SSOC.

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$$\begin{aligned}
H &= P^2(f_{11}f_{22} - f_{12}^2) \\
&= P^2[(\alpha-1)(\beta-1)\alpha\beta\, x_1^{2(\alpha-1)} x_2^{2(\beta-1)} - \alpha^2\beta^2\, x_1^{2(\alpha-1)} x_2^{2(\beta-1)}] \\
&= P^2[\alpha\beta\, x_1^{2(\alpha-1)} x_2^{2(\beta-1)}](1-\alpha-\beta)
\end{aligned}$$

The sign requirements on the main diagonal elements of H constrain both α and β to be less than one; the sign requirement on the full hessian determinant means that their sum must be less than one as well. The implications of these results also involve the notion of homogeneity.

■ HOMOGENEITY[3]

Homogeneity of a function is defined by the following equation; that is, if this equation is true for a specific function, then that function is homogeneous:

$$f(t\,x_1, t\,x_2, \ldots, t\,x_n) = t^r f(x_1, x_2, \ldots, x_n)$$

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where t is a multiplicative factor or scaler that represents a proportional change in all variables in the function and r is a constant that is called *the degree of homogeneity*. Essentially, a homogeneous function is one where a proportional increase in all variables can be factored out. As it is factored out, its effect on the level of the function is either heightened or dampened by the degree of homogeneity, that is, r is either greater or less than one.

The application to the Cobb-Douglas function is straightforward:

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[3] Silberberg (1990) section 3.7; Varian (1992) sections 1-11.

$$(t\,x_1)^{\alpha}(t\,x_2)^{\beta} \;=\; t^{\alpha+\beta}\,x_1^{\alpha}\,x_2^{\beta} \qquad\qquad 12$$

This shows that the C-D function is homogeneous of degree $\alpha + \beta$. The economic interpretation of a homogeneous production function can be summarized by reference to the degree of homogeneity. If the function is homogeneous of degree greater than one, then a *t*-fold increase in each input causes output to increase by more than *t*. This is commonly called "increasing returns to scale." [Note the use of the word scale.] If the degree of homogeneity is one, then we have constant returns to scale. In this case, doubling inputs, doubles output. If the degree of homogeneity is less than one, we have decreasing returns. Output increases by less than the increase in inputs.

An implication of the scale characterization of homogeneous production functions applies to the FOC of the profit maximization problem discussed above. The ratio of any two partial derivatives of a homogeneous function is itself homogeneous and of degree zero. This means that the ratio is invariant with respect to scale. That is exactly what we observed when we evaluated the FOC using the C-D production function. Now we find that this is general across all homogeneous functions. In economic terms, scale does not affect the choice of input mix for homogeneous functions. In this sense, for homogeneous functions we can talk about scale in the mathematical sense (a proportionate increase in all variables) as a valid economic concept because scale does not affect the economic choice of production mix.

Homogeneous functions are only homogeneous of a constant degree. That is, if a production function is homogeneous and exhibits increasing returns in some range, it will exhibit increasing returns in all ranges. (This means that functions that generate U-shaped average cost curves are not homogeneous.)

■ COMPARATIVE STATIC RESULTS[4]

Let's return to our general statement of the profit maximization problem and derive some comparative static results. The SSOC imply that the FOC can be solved for optimal values of the choice variables. In this case, satisfaction of the FOC and SSOC says that profit can be uniquely maximized by the choice of the levels of the two inputs.

When we have two (or more inputs), the FOC represent a system of equations that must be solved simultaneously. The SSOC ensure that this solution does in fact exist. Moreover, the solution of the FOC means that the optimal values of the choice variables will be expressed in terms of the things that were the parameters of the optimization problem.

For instance, the profit maximization problem has an economic agent maximizing profits by making the optimal choice of the levels of two inputs given the price of output and the price of the two inputs as constants. The prices are parameters in the optimization process.

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[4] Silberberg (1990) sections 6.1-6.2, 7.1-7.3

Once we know how this economic agent will act in the face of one set of parameters, we can determine how this behavior will change as these parameters change. The FOC and SSOC convert the choice variables in the optimization process to functions and parameters in the optimization problem to variables. In mathematical terms, the FOC and SSOC give us x_i^*'s that are values for the choice variables that optimize the objective function, and these x_i^*'s are themselves functions that can be written as

$$x_i^* = x_i^*(P, w_1, w_2), i = 1, 2 \qquad\qquad 13$$

Of course, it is these function in which we are interested. These are the functions that predict behavior. These functions tell us how the economic agent in this scientific model will logically behave when the test conditions of the model change.

When the model is derived using general functional representations of, for instance, the production function in the case at hand, the FOC cannot be solved explicitly for the x_i^* even though we know the conditions under which a solution is assured. This does not, however, mean that we have hit a dead end. Knowing that a solution exists, we can differentiate the FOC where they are evaluated at the solution values of the choice variables. We differentiate the FOC w.r.t. one (at a time) of the economic constraints in the optimization model.

That is, evaluate the FOC at the x_i^*:

$$14$$

$$P f_1(x_1^*(w_1, w_2, P), x_2^*(w_1, w_2, P)) - w_1 = 0$$
$$P f_2(x_1^*(w_1, w_2, P), x_2^*(w_1, w_2, P)) - w_2 = 0$$

Then differentiate w.r.t. one of the economic constraints, say, w_1:

$$15$$

$$P f_{11}(.) \frac{\partial x_1^*}{\partial w_1} + P f_{12}(.) \frac{\partial x_2^*}{\partial w_1} - \frac{\partial w_1}{\partial w_1} = 0$$

$$P f_{12}(.) \frac{\partial x_1^*}{\partial w_1} + P f_{22}(.) \frac{\partial x_2^*}{\partial w_1} = 0$$

In matrix form we have:

$$\begin{bmatrix} P f_{11}(.) & P f_{12}(.) \\ P f_{21}(.) & P f_{22}(.) \end{bmatrix} \begin{bmatrix} \dfrac{\partial x_1^*}{\partial w_1} \\ \dfrac{\partial x_2^*}{\partial w_1} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad\qquad 16$$

• Let's write the matrix representation of the comparative static analysis shown in eqt. (16) as *A x = y*. We immediately recognize that *A* is a matrix composed of the elements of the hessian determinant which comprises the SSOC. In that sense, *A* is a matrix made up of the second

derivative of the objective function w.r.t. the choice variables of the optimization problem. The matrix *x*, which in this case is a vector, is made up of the partials of the behavioral equations, the x_i^*, w.r.t. the parameter of interest from the optimization problem. Finally, we have the matrix *y*. In this case *y* is also a vector of the same rank as *x*. The elements in *y* are the second, cross partials of the objective function w.r.t. the choice variables and w.r.t. the parameters.

From matrix algebra, we know that *x* has a solution if we can write $x = A^{-1}y$. This we can do if $|A|$ is not equal to zero. Because $|A|$ is the hessian determinant *H* that is defined as nonzero by the SSOC, we know that our problem can be solved for the terms in the vector *x*, i.e., the comparative static solutions.

• Solving for the comparative static solutions in the problem at hand, we have the following based on Cramer's rule:

$$\frac{\partial x_1^*}{\partial w_1} = \frac{\begin{vmatrix} 1 & P f_{12}(.) \\ 0 & P f_{22}(.) \end{vmatrix}}{\begin{vmatrix} P f_{11}(.) & P f_{12}(.) \\ P f_{21}(.) & P f_{22}(.) \end{vmatrix}} = \frac{1}{P} \frac{f_{22}}{f_{11} f_{22} - f_{12}^2} \qquad 17$$

and

$$\frac{\partial x_2^*}{\partial w_1} = \frac{\begin{vmatrix} P f_{11}(.) & 1 \\ P f_{21}(.) & 0 \end{vmatrix}}{\begin{vmatrix} P f_{11}(.) & P f_{12}(.) \\ P f_{21}(.) & P f_{22}(.) \end{vmatrix}} = \frac{1}{P} \frac{- f_{12}}{f_{11} f_{22} - f_{12}^2} \qquad 18$$

We know that the center expression in eqt (17) is negative because of the SSOC. The denominator is the full hessian determinant; the numerator is the principal minor of the full hessian determinant one order lower. By the SSOC, these must alternate in sign. Hence, this result will clearly hold in general, that is, in the *n* input case. If an input's price increases, its usage must decline. More specifically in this case we know that f_{22} must be negative and $f_{22}f_{11} - f_{12}^2$ positive by the SSOC.

While we cannot say anything generally about eqt. (18) because the SSOC do not demand that f_{12} be signed, in the case of the two input C-D production function, where the SSOC are satisfied, we know that f_{12} is, in fact, positive. Hence, we can say that for this production function, an increase in one input price will cause the usage of the other input to fall as well.

It is important to recognize that this is the predicted behavior for a profit maximizing firm as it adjusts to a change in input prices holding all other parameters constant, most importantly the price of output. In this sense the model is a partial equilibrium analysis. The demand curves that are derived from this model do not account for the output market equilibrium that will change as a consequence of the change in an input price. We will elaborate on this result at length later.

• It is somewhat interesting to develop the comparative static analysis of the behavior of the firm in response to a change in the price of output. Following from eqt. (19) above, we can write:

$$\begin{bmatrix} P f_{11}(.) & P f_{12}(.) \\ P f_{21}(.) & P f_{22}(.) \end{bmatrix} \begin{bmatrix} \dfrac{\partial x_1^*}{\partial P} \\ \dfrac{\partial x_2^*}{\partial P} \end{bmatrix} = \begin{bmatrix} -f_1 \\ -f_2 \end{bmatrix}$$

This can be solved simply enough for the response rates of the two inputs w.r.t. the price of output. Note, however, that these cannot be signed without making an assumption about the value of f_{12}. Even so, we can remark on the following:

In order for $\dfrac{\partial x_1^*}{\partial P} < 0$, $-f_2 f_{12} > -f_1 f_{22}$. Similarly, in order for $\dfrac{\partial x_2^*}{\partial P} < 0$, $-f_1 f_{12} > -f_2 f_{11}$. However, in order for both inputs to respond negatively to an increase in the price of output, $f_1 f_2 f_{12}^2 > f_1 f_2 f_{11} f_{22}$ which cannot be true if the SSOC hold. An increase in the price of output must cause at least one input to increase in use.

Questions:

Use eqt (9) to show that Cobb-Douglas demand curves are downward sloping.

Use eqt (9) to determine the effect on x_1 of an increase in P.

Discuss the implications of the SSOC on α and β in the C-D production function.