Transcendental Logarithmic Cost Function[1]

The purpose of the translog cost function is to identify a specific functional form for a cost function that embodies all of the assumptions and results of our cost minimization model. The importance of a specific functional form is that it can be used in empirical work. In particular, we want a cost function that allows for U-shaped average cost. The other conditions include the following:

1. Input demand is downward sloping.
2. Cross price effects are symmetric.
3. The shift in marginal cost w.r.t. an input price is equal to the shift in the input's demand w.r.t. output.
4. The sum of own and cross price elasticities is equal to zero.
5. A proportional increase in all input price must shift cost by the same amount holding output constant.

The translog function looks like the following:

$$\ln C^* = \alpha_0 + \alpha_q \ln q + \frac{1}{2}\gamma_{qq}(\ln q)^2 + \sum_i \gamma_{qi} \ln q \ln w_i$$
$$+ \sum_i \alpha_i \ln w_i + \frac{1}{2}\sum_i \sum_j \gamma_{ij} \ln w_i \ln w_j$$

We know from our work on cost functions that the envelope theorem implies that $\partial C^* / \partial w_i$ is equal to x_i^*. In logarithmic form $\frac{\partial \ln C^*}{\partial \ln w_i} = \frac{\partial C^*}{\partial w_i}\frac{w_i}{C^*}$. Substituting the result from the envelope theorem, we have $\frac{\partial \ln C^*}{\partial \ln w_i} = x_i^* \frac{w_i}{C^*} = S_i^*$, where S_i^* is the cost share of the *i*th input. From the translog cost function we have

$$\frac{\partial \ln C^*}{\partial \ln w_i} = S_i^* = \alpha_i + \gamma_{qi} \ln q + \sum_j \gamma_{ij} \ln w_j$$

where $\gamma_{ij} = \gamma_{ji}$.[2]

The translog cost function is estimated as a system of equations. The aspects of the firm's behavior that we observe are total cost, the allocation of total cost across the various inputs (i.e., input expenditure shares), the firm's output level, *q*, and the input prices that the firm faces.

One restriction on the parameter estimates across equations is that imposed by linear homogeneity of the cost function w.r.t. input prices. That is, a proportional increase in all input

[1] See L. Christensen and W. Greene, "Economies of Scale in U.S. Electric Power Generation," *JPE*, 84(4), 76, 655-676. Nicholson (1998) Ch 12 and appendix. Varian (1992) Sections 12.17 - 12.10.

[2] Note how the double sum and the {1/2} term cancel out. The *ii* term in the double sum is squared. Its derivative cancels the one-half. And, since there are two *ij* terms that have identical coefficients, the one-half cancels there as well.

prices must increase cost by the same proportion, holding output constant. If we take the total differential of the log of cost, holding output constant, we get the following:

$$d\ln C^* = \sum_i [\gamma_{qi} \ln q]\, d\ln w_i + \sum_i \alpha_i \, d\ln w_i + \frac{1}{2}\sum_i \sum_j \gamma_{ij}\, d\ln w_i \, d\ln w_j$$

By assumption, $d\ln w_i$ is equal across all n inputs. Hence we can factor these out. This gives

$$d\ln C^* = d\ln \overline{w} \sum_i [\gamma_{qi} \ln q] \; + d\ln \overline{w} \sum_i \alpha_i \; + d\ln \overline{w}^2 \, \frac{1}{2}\sum_i \sum_j \gamma_{ij}$$

In order for $\frac{d\ln C^*}{d\ln \overline{w}} = 1$, the following constraints on the parameters must hold:

$$\sum_i \alpha_i = 1$$
$$\sum_i \gamma_{qi} = 0$$
$$\sum_i \sum_j \gamma_{ij} = 0$$

Of particular interest is the scale economy effect. The translog function allows for both positive and negative scale effects, that is, average cost can both decrease and increase across the range of the cost function. In this sense, the translog function can represent a production function that is not homogeneous. A standard expression is to write scale effects as $\left[1 - \partial \ln C^* / \partial \ln q\right]$. If optimized cost rises slower than output, the scale effect is positive implying a declining average cost function. The expression depends on the elasticity of cost w.r.t. output, which is given by

$$\frac{\partial \ln C^*}{\partial \ln q} = \alpha_q + \gamma_{qq} \ln q + \sum_i \gamma_{qi} \ln w_i$$

The elasticity of cost w.r.t. output is the ratio of marginal to average cost. If marginal is above average, average is rising; and if marginal lies below, average is falling.

$$\frac{\partial \ln C^*}{\partial \ln q} = \frac{\partial C^*}{\partial q}\frac{q}{C^*} = \frac{MC}{AC}$$

From the trans-log specification, the change in this elasticity as output changes is given by γ_{qq}. If this parameter is positive, then any scale effect that exists will run out at sufficiently high output levels. If the other parameters are such that the output elasticity of cost is less than one at low output levels, then we have a U-shaped average cost function.

Other results can be identified from this expression. Homotheticity of the underlying production function requires that $\gamma_{qi} = 0$ for all i. Homogeneity requires that $\gamma_{qq} = 0$ as well. Note that the homotheticity and homogeneity results are not restrictions on the estimating form. They are empirical observations that we can draw from this function when it is estimated. If $\gamma_{qi} = 0$ for all *i*, then the underlying production process is homothetic; if $\gamma_{qq} = 0$, then it is homogeneous as well. Recall from the discussion of the Cobb-Douglas form that the ratio of marginal to average cost is a constant. That is, if average cost is rising, marginal cost is always above it by a constant proportion. With constant returns, the ratio is one, and with increasing returns, marginal cost is a constant fraction of average. In the translog function, if γ_{qq} and the γ_{qi}'s are zero, we get this same result:

$$\frac{\partial \ln C^*}{\partial \ln q} = \alpha_q = \frac{\partial C^*}{\partial q}\frac{q}{C^*} = \frac{MC}{AC}$$

The translog function allows for estimation of parameters that embody all of the relations that are derivable from the general model of cost minimization subject to a output objective. In particular we can show the following:

$$\frac{\partial x_i}{\partial w_i}\frac{w_i}{x_i} = \frac{\gamma_{ii}}{S_i} + S_i - 1$$

In words, own-price elasticity of input demand is given by an expression including the estimated parameter γ_{ii}. From this expression we see that downward sloping demand requires that γ_{ii} be less than the input's cost share times one minus its cost share. This form is derived in the following fashion

$$S_i^* = \frac{w_i x_i^*}{C^*} \quad \text{so} \quad x_i^* = C^* S_i^* w_i^{-1}$$

$$\frac{\partial x_i^*}{\partial w_i} = \frac{\partial C^*}{\partial w_i} S_i^* w_i^{-1} + \frac{\partial S_i^*}{\partial w_i} C^* w_i^{-1} - w_i^{-2} C^* S_i^*$$

$$= x_i^* S_i^* w_i^{-1} + \frac{\gamma_{ii}}{w_i^*} C^* w_i^{-1} - w_i^{-2} C^* S_i^*$$

In a similar fashion we can derive the cross-price effects:

$$\frac{\partial x_i^*}{\partial w_j} = \frac{\partial C^*}{\partial w_j} S_i^* w_i^{-1} + \frac{\partial S_i^*}{\partial w_j} C^* w_i^{-1}$$

$$= x_j^* S_i^* w_i^{-1} + \frac{\gamma_{ij}}{w_j^*} C^* w_i^{-1}$$

$$= \frac{x_i x_j}{C^*} + \gamma_{ij} \frac{C^*}{w_i w_j}$$

From this expression it can be shown that the cross-price effects are symmetric and that the cross-price elasticity is equal to $S_j^* + \frac{\gamma_{ij}}{S_i^*}$. The restriction, $\sum_j \gamma_{ij} = 0$, ensures that the sum of own and cross price elasticities is zero. Symmetry of the cross price effects requires that $\gamma_{ij}=\gamma_{ji}$, which is a restriction in the estimation.

Finally it can be shown that the translog cost function obeys the symmetry result that says the change in marginal cost w.r.t. an input price is equal to the change in input demand w.r.t. output. We start with the identity:

$$\frac{\partial \ln C^*}{\partial \ln q} = \frac{\partial C^*}{\partial q}\frac{q}{C^*} = \frac{MC}{AC}$$

Thus, we can write:

$$\frac{\partial MC}{\partial w_i} = \frac{\partial \left[\frac{\partial \ln C^*}{\partial \ln q} AC\right]}{\partial w_i} = \frac{\partial^2 \ln C^*}{\partial \ln q \partial w_i} AC + \frac{\partial \ln C^*}{\partial \ln q}\frac{\partial AC}{\partial w_i} = \frac{\gamma_{qi}}{w_i} AC + \frac{\partial \ln C^*}{\partial \ln q}\frac{x_i}{q}$$

Working back the other way on the input demand we see that

$$\frac{\partial x_i^*}{\partial q} = \frac{\partial C^*}{\partial q} S_i^* w_i^{-1} + \frac{\partial S_i^*}{\partial q} C^* w_i^{-1} = \frac{\partial C^*}{\partial q}\frac{x_i^*}{C^*} + \frac{\gamma_{qi}}{q} C^* w_i^{-1} = \frac{\partial \ln C^*}{\partial \ln q}\frac{x_i^*}{q} + \frac{\gamma_{qi}}{w_i} AC$$

The translog cost function and cost share equations are estimated as a system. In the estimation, the restrictions on the coefficients are imposed across equations. Essentially, this allows the observable information about the behavior of the firm (total resources expenditures, the distribution of these expenditures across inputs, and the output yielded by these expenditures) and the resources prices faced by the firm all to be used in the estimation of the parameters of the model.

In addition to the restrictions on the estimated parameters, there is the restriction that the error terms in the share equations sum to zero.

Cost Shares

Consider the circumstances under which the cost share of an input can increase as the input's price increases. First we write share:

$$S_i^* = \frac{x_i^* w_i}{C^*}$$

Then take the log and the derivative w.r.t. the input's own price:

$$\ln S_i^* = \ln x_i^* + \ln w_i - \ln C^*$$

$$\frac{\partial \ln S_i^*}{\partial \ln w_i} = \frac{\partial \ln x_i^*}{\partial \ln w_i} + 1 - \frac{\partial \ln C^*}{\partial \ln w_i} >< 0$$

Rewriting gives:

$$\frac{\partial \ln S_i^*}{\partial \ln w_i} = \varepsilon_{ii} + (1 - S_i^*) >< 0$$

If this is positive, share increases with price. It will be positive if input demand is inelastic and strongly so. In order to obtain such a result, input demand must be very inelastic if the input's share of cost is large.

For the Cobb-Douglas production function, own-price elasticity of input demands is constant and equal to one minus the cost share, so the cost share never changes.

Symmetrically, if the cost share is constant, then own-price elasticity must equal one minus the cost share.

OUTLINE & SUMMARY OF PRODUCTION THEORY

I. BUILDING BLOCKS

A. Definition of short and long run

1. Definitions are vague. Terms are used to reflect many circumstances not all of which are time related.
2. Short run implies one (or maybe more) of the following: some inputs are fixed for the firm; the level of some resources fixed for the industry; the number of firms in the industry is fixed; industry price is not an equilibrium value.
3. Long run in the limit means that everything is variable and all values are equilibrium. This means the size of the firms, the resources employed by firms, the number of firms, and prices in input and output markets.

B. Model of Cost Minimization subject to an Output Objective

1. Marginal cost and input demands are the FOC
2. Total and average cost are implied
3. Applies to all market structures
4. U-shaped AC gives determinate firm size

C. Implications of the model

1. Input demand is downward sloping.
2. Input demand curves are symmetric.
   a) Cross price effects are symmetric.
   b) The shift in marginal cost w.r.t. an input price is equal to the shift in the input's demand w.r.t. output.
3. Input demand curves are homogeneous of degree zero with respect to input prices.
   a) The sum of own and cross price elasticities is equal to zero.
   b) A proportional increase in all input prices
      (1) Must shift cost by the same amount holding output constant. (This means that average and marginal cost shift vertically by the factor of proportion.)
4. The magnitude of the effect of input price on average cost
   a) Equal to the ratio of input usage to output (envelope result).

II. FIRM IN COMPETITIVE MARKET

A. Output price is exogenous

1. Per firm demand curve
   a) Does not imply pricing power

B. Simple profit maximization model

1. Gives relations among optimal input choices
2. Value of Marginal Product (VMP) equals Input Price

III. FIRM AT THE INDUSTRY EQUILIBRIUM

A. Some illustrations from government regulation and taxation

a) Quotas and input restrictions
   (1) Transferability of quotas
   (2) Per unit taxes compared to business license
b) Transferability of input restrictions

Transferable input usage rights are same as market in the input. The result on average and marginal cost is same as if supply of the actual input shifted.
Define input demand and input demand elasticity.

IV. INDUSTRY SUPPLY

A. Industry returns to scale

1. Assume Constant Returns to Scale (CRS)
2. Input shares for CRS
   a) Euler's Theorem and VMP input usage
      (1) Shares (S_i) sum to one
      (2) Shares are equal to elasticity of output with respect to input

B. Changes in Equilibrium

1. Input price changes
   Define industry input demand as industry responds to price change
   b) Demand elasticity holding output constant ("short run")
   $$\varepsilon_{11}^* + \varepsilon_{12}^* = 0$$
   Define elasticity of substitution: $\sigma = \varepsilon_{12}^* / S_2$.
   $$\varepsilon_{11}^* = -(1 - S_1)\sigma$$
   c) Demand elasticity at industry equilibrium in output market ("long run")
   $$\varepsilon_{11}^{**} = \varepsilon_{11}^* + \eta S_1$$
   $$\varepsilon_{11}^{**} = -(1 - S_1)\sigma + S_1\eta$$
   d) Inputs as complements and substitutes
   $$\varepsilon_{12}^{**} = S_2(\sigma + \eta)$$
   e) Cobb-Douglas
   σ=1
   f) Marginal Cost
   $$\frac{\partial \ln MC}{\partial \ln w_1} = \frac{\partial MC}{\partial w_1}\frac{w_1}{MC} = \frac{x_1^* w_1}{Q \cdot MC} = S_1$$
   g) Shares
   $$\ln S_1 = \ln w_1 + \ln x_1^* - \ln C^* \text{ (short run)}$$
   $$\ln S_1 = \ln w_1 + \ln x_1^{**} - \ln MC - \ln Q^* \text{ (long run)}$$
   $$\frac{\partial \ln S_1}{\partial \ln w_1} = 1 + \varepsilon_{11}^{**} - S_1(1+\eta) = 1 + \varepsilon_{11}^* - S_1$$
2. Output demand changes
   a) Shape of industry input supply implies industry output supply
   As industry output demand increases, input demand increases.
   The equilibrium input price change as a result of this shift affects MC.
   If input supply is positively sloped, input price increases, which shifts MC up. This is the case of a positively sloped industry supply function.
   b) Industry output supply elasticity
   Equals the elasticity of MC with respect to input price times the elasticity of equilibrium input price with respect to output
   $$1/\eta^S = \frac{\partial \ln MC}{\partial \ln w_1}\frac{\partial \ln w_1^*}{\partial \ln Q} + \frac{\partial \ln MC}{\partial \ln w_2}\frac{\partial \ln w_2^*}{\partial \ln Q} = S_1 \cdot \frac{1}{\lambda_1 - \varepsilon_{11}^*} + S_2 \cdot \frac{1}{\lambda_2 - \varepsilon_{22}^*}$$
3. Marshall's Last Law
   a) Input demand elasticity is greater the greater is the elasticity of supply of other inputs.