ODDS AND ENDS

Cobb-Douglas Production & Profit Maximization[1]

The problem is to assess the restrictions that the SSOC for profit maximization place on the parameters of the standard Cobb-Douglas production function. Let's start out with the general form assuming downward sloping demand:

$$\max \Pi = p(f(x_1, x_2))\, f(x_1, x_2) - w_1 x_1 - w_2 x_2 \qquad \mathbf{1}$$

The FOC have the form:

$$p' f_i f + p f_i - w_i = 0 \qquad \mathbf{2}$$

The main diagonal elements of the hessian determinant defining the SSOC have the following form:

$$p'' f_i^2 f + p' f_{ii} f + 2p' f_i f_i + p f_{ii} \qquad \mathbf{3}$$

Rewriting gives:

$$f_i^2 (p''f + 2p') + f_{ii}(p'f + p) \qquad \mathbf{4}$$

The off-diagonal elements look like:

$$p'' f_2 f_1 f + p' f_{12} f + p' f_1 f_2 + p' f_2 f_1 + p f_{12} \qquad \mathbf{5}$$

Rewriting gives:

$$f_1 f_2 (p''f + 2p') + f_{12}(p'f + p) \qquad \mathbf{6}$$

Multiplying out the hessian, we have:

$$(p''f + 2p')(p'f + p)[f_{11} f_2^2 + f_{22} f_1^2 - 2 f_{12} f_1 f_2] + (p'f + p)^2 [f_{11} f_{22} - f_{12}^2] > 0 \qquad \mathbf{7}$$

The Cobb-Douglas production function gives us the terms to substitute into the SSOC.

$$f_1 = \frac{\partial q}{\partial x_1} = \alpha q x_1^{-1}; \quad f_{11} = \frac{\partial^2 q}{\partial x_1^2} = \alpha(\alpha - 1) q x_1^{-2}; \quad f_{12} = \frac{\partial^2 q}{\partial x_1 \partial x_2} = \alpha\beta q x_1^{-1} x_2^{-1} \qquad \mathbf{8}$$

Note that the first two terms in eqt. (7) are the slope of marginal revenue and marginal revenue, which we label as *R*?? and *R*?, respectively. In the competitive case marginal revenue is price and has a slope of zero.

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[1]Silberber (1990) Ch 6

First we look at the full hessian.

$$R'R''\alpha\beta q^3 x_1^{-2} x_2^{-2}(-\beta-\alpha) - (R')^2\alpha\beta q^2 x_1^{-2} x_2^{-2}(1-\alpha-\beta) > 0 \qquad \textbf{9}$$

Canceling, we have:

$$-R''q(\alpha+\beta) + R'(1-\alpha-\beta) > 0 \qquad \textbf{10}$$

Rewriting gives:

$$-R'' > \left(\frac{R'}{q}\right)\left(1-\frac{1}{\alpha+\beta}\right) \qquad \textbf{11}$$

This says that if $R'' < 0$ and $\alpha + \beta < 1$ then the SSOC are satisfied. In the interesting case, if $\alpha + \beta > 1$, then *R'' must be negative and big enough to satisfy eqt.* (11).

Next, we examine the condition imposed by the principal minors. Substitute from eqt.(8) into eqt. (4):

$$R''\alpha^2 q^2 x_1^{-2} + R'\alpha(\alpha-1)q\, x_1^{-2} < 0 \qquad \textbf{12}$$

Canceling and rewriting gives:

$$-R'' > \left(1-\frac{1}{\alpha}\right)\left(\frac{R'}{q}\right) \qquad \textbf{13}$$

Similar to eqt. (11), eqt. (13) says that if α alone is greater than 1, then marginal revenue must be downward sloping and relatively large in absolute terms.

The Comparative Statics of 3rd Degree Price Discrimination: The Common Cost Problem[2]

Intel produces the microprocessors for personal computers. The top end of the market for a while was satisfied by a processor called the 486. Intel marketed two different chips—The regular 486 and the 486SX. The SX is just a slower version and is priced a good deal less. Assume that cost is identical for both chips. How do they set the prices? What happened to the relative prices as cost declined?

[2]Varian (1992) Ch 14 (Section (14-8); Layard & Walters (1978) Section 8.1; Nicholson (1998) Ch 18 (562-565).

The standard approach goes as follows: The firm maximizes profits across the two products:

$$\max \Pi = R_1(q_1) + R_2(q_2) - C(q_1 + q_2) \qquad 14$$

FOC:

$$\Pi_i = R_1' - C' = 0, i = 1,2$$

SSOC:

$$\begin{vmatrix} R_1'' - C'' & -C'' \\ -C'' & R_2'' - C'' \end{vmatrix} > 0 \qquad 15$$

Let cost shift with respect to some parameter α. That is, differentiate the FOC w.r.t. a parameter α; this gives an expression C_a in the vector of constants. Solve for the comparative static expressions for $dq_1/d\alpha$ and $dq_2/d\alpha$.

The question asks you to solve for the change in the price ratio as cost shifts. This is given by:

$$\frac{d\left(\frac{P_1}{P_2}\right)}{d\alpha} = \frac{\frac{dP_1}{d\alpha}P_2 - \frac{dP_2}{d\alpha}P_1}{P_2^2} \qquad 16$$

where

$$\frac{d\ P_i}{d\alpha} = \frac{d\ P_i}{dq_i}\frac{dq_i}{d\alpha} \qquad 17$$

The expression dP_i/dq_i is the slope of the demand curve, and we can substitute for $dq_1/d\alpha$ and $dq_2/d\alpha$ from the standard comparative static analysis:

$$\begin{bmatrix} R_1'' - C'' & -C'' \\ -C'' & R_2'' - C'' \end{bmatrix} \begin{bmatrix} \frac{\partial q_1}{\partial \alpha} \\ \frac{\partial q_2}{\partial \alpha} \end{bmatrix} = \begin{bmatrix} C_\alpha \\ C_\alpha \end{bmatrix}$$

From this we can write:

$$\frac{d\left(\frac{P_1}{P_2}\right)}{d\alpha} = \frac{C_\alpha (P_1' R_2'' P_2 - P_2' R_1'' P_1)}{P_2^2 H}$$

It is best to set this expression up as a greater-than-less-than-zero inequality. In that form the denominator expressions (P_2^2 and the SSOC term) cancel out as does the term C_α. We are left with:

$$\frac{d(P_1 / P_2)}{d\alpha} < 0 \text{ as } P_1' R_2'' P_2 - P_2' R_1'' P_1 < 0 \qquad 18$$

or

$$\frac{d\left(\frac{P_1}{P_2}\right)}{d\alpha} < 0 \text{ as } \frac{P_1}{P_2} > \frac{\left(\frac{P_1'}{R_1''}\right)}{\left(\frac{P_2'}{R_2''}\right)} \qquad 19$$

Note the inequality reversal. I am implicitly assuming that both demand and marginal revenue are downward sloping.

Eqt. (6) is best evaluated in the context of specific functional form for demand. There are three common forms: 1) linear; 2) semi-log; 3) double-log. Let's examine each in turn. The important features of the function form are the slope of demand and slope of marginal revenue.

Linear:

$$P = a + bq \text{ and } Pq = aq + bq^2 \qquad 20$$

$$P' = b, \quad R' = a + 2bq \text{ and } R'' = 2b \qquad 21$$

In this case, the ratio of the slope of demand to the slope of marginal revenue is always one-half. For any two demands that are both linear, the right hand side of eqt. (6) is always 1. Hence, if $P_1/P_2 > 1$, that is, if demand 1 is the less elastic, higher priced market, then $d(P_1/P_2)/d\alpha < 0$. This says that as costs fall the price ratio diverges.

Semi-Log:

$$P = \alpha + \beta \ln q \text{ and } Pq = [\alpha + \beta \ln q]q \qquad 22$$

$$P' = \frac{\beta}{q}, \quad R' = P + \beta \text{ and } R'' = \frac{\beta}{q} \qquad 23$$

In this case, the ratio of the slope of demand to the slope of marginal revenue is always one. For any two demands that are both semi-log, the right hand side of eqt. (6) is always 1. Hence, if $P_1/P_2 > 1$, that is, if demand 1 is the less elastic, higher priced market, then $d(P_1/P_2)/d\alpha < 0$; again, as costs fall the price ratio diverges.

Double-Log:

$$P = Aq^B \text{ and } Pq = Aq^{B+1} \qquad 24$$

$$P' = BAq^{B-1}, \quad R' = (B+1)Aq^B \text{ and } R'' = B(B+1)Aq^{B-1} \qquad 25$$

In this case, the problem is a little more complex. First, the SSOC are only satisfied when B is less than 1 in absolute value. B is the inverse of price elasticity. The SSOC says that if demand is inelastic, then there is no profit maximum. The monopolist raise price to infinity. When the SSOC is satisfied by B falling in the appropriate range, the FOC define the price ratio:

$$\frac{P_1}{P_2} = \frac{B_2+1}{B_1+1} \qquad 26$$

Likewise

$$\frac{P_1'/R_1''}{P_2'/R_2''} = \frac{B_2+1}{B_1+1} \qquad 27$$

Hence, in the double log case eqt. (19) is equal to zero. That is,

$$\frac{d\left(\frac{P_1}{P_2}\right)}{d\alpha} = 0 \text{ since } \frac{P_1}{P_2} = \frac{P_1'/R_1''}{P_2'/R_2''} \qquad 28$$

As a final thought, consider the comparative statics of the firm that sells competitively in one market and monopolistically in another. This should be similar to intrafirm trading versus outside sales.

Cournot Model of Oligopoly Industry Equilibrium[3]

Consider the simple profit maximization model:

$$\pi_i = P(Q)q_i - C_i(q_i)$$

where Q is industry output, i.e., the sum of the q_i. This says that the i^{th} firm in the industry faces a demand curve that is a function of the total industry output. The question is, How does price change as the output of the i^{th} firm changes? This can be developed by looking at the FOC for profit maximization:[4]

$$\frac{\partial \pi_i}{\partial q_i} = P'(Q)\cdot\left[\frac{\partial Q}{\partial q_i}+1\right]\cdot q_i + P - C_i' = 0 \qquad 29$$

The term $\partial Q/\partial q_i$ is the key. Under competitive assumptions, $\partial Q/\partial q_i$ is –1. This means that any output reduction by one competitor is matched by another. The first term cancels out and the competitive result, $P=MC$, prevails. The Cournot model assumes that $\partial Q/\partial q_i$ is zero. Fama and Laffer wittily characterize the Cournot model by saying that Cournot firms mind their p's but not their q's. Game theorists latched onto this form and assign various values to $\partial Q/\partial q_i$ between zero and –1. This approach is called conjectural variations.

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[3] Varian (1992) section 16.1-16.3; Layard and Walters (1978) section 19.4; Nicholson (1998)Ch 19, p 580-592; Silberberg (1990) section 19.4.

[4] See Eugene Fama and A.B. Laffer, "The Number of Firms and Competition," *AER*, Sept 72, 670-674.

The Cournot model, based on the assumption that $\partial Q/\partial q_i = 0$, assumes that each firm expects that all other firms will hold output constant. That is, firm one believes that when it changes output, firms two, three, etc., will not react to its output adjustment. However, other firms do not do this. However, when they respond to changes in output by firm one, they, too, assume that firm one and the other firms will hold the line on output. This is why Fama and Laffer say, “Cournot firms mind their p’s but not their q’s.”

Having said this, the Cournot assumption is still not particularly revealing in terms of the market equilibrium. What the Cournot model says is that each firm in the industry behaves according to a FOC given by equation (1) under the assumption that $\partial Q/\partial q_i = 0$. This means that market price and output is found by solving equation (1) simultaneously for the firms in the industry. In order to more clearly see what the Cournot solution looks like, it is useful to consider some specific functional forms.

Assume that there are three firms. Let the cost function of each firm be:

$$C_i = q_i^2 \qquad 30$$

Let the demand function be:

$$P = 100 - (q_1 + q_2 + q_3) \qquad 31$$

The profit function of the first firm is:

$$\pi_1 = R_1 - C_1 = [100q_1 - q_1^2 - q_1q_2 - q_1q_3] - q_1^2$$

The FOC for profit max is:

$$\frac{\partial \pi_1}{\partial q_1} = 100 - 4q_1 - q_2 - q_3 = 0 \qquad 32$$

The other firms look exactly the same.

The equilibrium solution in the Cournot oligopoly given these specific functional forms is found by the simultaneous solution of the FOC’s (given by equation 4) of the firms in the industry. The equilibrium of this model is said to be characterized by *reaction functions*; the FOC’s embodying the Cournot assumption about $\partial Q/\partial q_i$ represent what are called the reaction functions.

The simultaneous solution is most easily depicted in matrix form:

$$\begin{bmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 100 \\ 100 \\ 100 \end{bmatrix}$$

For the specific demand and cost functions that we have chosen, the solution for the output of each firm is, $q_i = 16.667$, and industry output is 50. Substituting these back into the demand and cost functions gives: P=50 and average profits, π_i / q_i, equal 33.33. Average profits are just price minus average cost. From equation (2), average cost is equal to q.

The Cournot model is quite often presented where the firms have zero marginal cost. In this form, there is a very simple relationship between the competitive output and the Cournot equilibrium. Cournot oligopolists produce [n/(n+1)] of the competitive output. Notice that when an upward sloping marginal cost function is used, you don't get the simple n/(n+1) relation between the oligopoly output and competitive output. When production at the firm level is characterized by the cost function given by equation (2), the competitive equilibrium will not be characterized by zero profit unless there are an infinite number of firms. In the case where marginal cost is zero (or just flat), the competitive equilibrium obtains with any number of firms. Hence, the comparison between Cournot oligopoly and competition is more meaningful than if marginal cost is positively sloped.

The cost function in (2) says that costs approach zero as output gets smaller and smaller. This means that the competitive equilibrium is characterized by a *de minimus* scale of production for each competitor. As the number of firms gets larger and larger, price approaches zero. Hence, we can say that the competitive market long-run output is approximately 100.

For any given number of firms, we can define the competitive equilibrium by setting price equal to marginal cost. Market price is, $P = 100 - nq$. Marginal cost is, $MC = 2q$. By equating price and marginal cost, we can solve for firm level output, which is [100/(n+2)]. From this we can solve for market price and output. Competitive output with two firms is 50; with three firms, 66.7; with four firms, 75.

Given the cost function in equation (2), a Cournot duopoly has industry output of 40; three firms yields 50; four firms gives 57. Whether compared to the long-run or short-run competitive market, with a positively sloping marginal cost function, there is no simple relation between the Cournot equilibrium and competition except to say that the Cournot industry output is always smaller.