Discounting Problems[1]

■ DERIVATION of the Natural Number

The introduction of the operator *ln*, that is, the natural log, raises the question of the natural number. The natural number comes from the concept of discounting.

• Suppose that you have \$1 in principal and earn x% annual interest, then you will have $y = 1 + x$ at the end of the year if the interest is compounded once.

If interest is compounded twice, $y = 1 + x/2 + (1+x/2)x/2$. [$x/2$] are the interest earnings at the first compounding, yielding a principal at that time of [$1+x/2$]. [$(1+x/2)x/2$] are the interest earnings over the second half of the year. This semi-annual compounding can be reduced to $y=(1+x/2)^2$. In similar fashion, quarterly compounding can be shown to reduce to $y=(1+x/4)^4$.

In general, then, $y=(1+x/n)^n$ is the expression for future value when x is the annual interest rate and compounding occurs n times per year. The natural number, e, results from letting n in this expression go to infinity. That is, e answers the question of what happens when the compound rate of growth is continuous.

• To develop the expression, let $x = 1$.

$$y = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \;=\; 1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\ldots \qquad 1$$

The right-most expression in equation (30) is the expansion of the limit by the Binomial Theorem. The Binomial Theorem is

$$(a+b) \;=\; \sum_{j=0}^{n}\left(\frac{n!}{j!(n-j)!}\right) a^j \; b^{n-j} \qquad 2$$

The application of the theorem is fairly straightforward. Let $a = 1/n$, and $b = 1$. The higher order terms have components that look like $(n-k)/n$ which can be rewritten to $(1-k/n)$. As n goes to infinity, these terms go to 1, which results in the expression shown in eqt.(30).

When the infinite series is evaluated, to five decimal places it is 2.71828 and it is called e.

• In general, let x take any positive value.

$$y \;=\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n \qquad 3$$

[1] Bentick, Brian L. "The Impact of Taxation and Valuation: Practices on Timing and Efficiency of Land Use" in Journal of Political Economy Vol. 87, 1979. p 859-68
Nicholson (1998) Ch 18 p. 557-559, p. 700-708.

Using a trick substitution where $m=n/x$ we can write:

$$y = \left(\lim_{n\to\infty}\left(1+\frac{1}{m}\right)^{m}\right)^{x} = e^{x} \qquad 4$$

Note also using the expansion by the Binomial Theorem that

$$e^{x} = 1+\frac{x}{1!}+\frac{x^{2}}{2!}+\frac{x^{3}}{3!}+\ldots \qquad 5$$

Thus, the derivative is

$$\frac{de^{x}}{dx} = 1+\frac{x}{1!}+\frac{x^{2}}{2!}+\frac{x^{3}}{3!}+\ldots = e^{x} \qquad 6$$

• For principal values different from $1 we can write

$$FV = PVe^{rt} \qquad 7$$

where PV is the initial principal, r is the interest rate (note the notation change just to make this expression more common), and FV is the future or compound value after t periods. If r is stated in annual terms, then t is measured in years.

The function inverts directly.

$$PV = FVe^{-rt} \qquad 8$$

The inverse of future value is present value.

■ OPTIMIZING BEHAVIOR[2,3]

A handful of interesting problems solved rather easily using the Natural Number. The first is the Fisher Investment Problem or "When do you cut the tree?" Let $g(t)$ describe the growth in value of an asset like a tree or a bottle of wine. Assume that the function is positively sloped and concave. Value grows at a decreasing rate. At any point in time, the value can be capitalized, by cutting the tree, or it can be allowed to continue to grow. The function $g(t)$ describes the future value of the investment at each point in time. The question is what is the optimal length of time to allow the investment to continue.

[2] Review of above material in Silberberg (1990), section 2.9, part 1.
[3] Nicholson (1998) Chapter 2. Review of optimization/maxization; simple calculus with examples (good general review. Chapter 23, pages 703-708.

This can be answered by maximizing the present value of the investment over the choice of *t*.

$$\max_{\{t\}} PV(t) = g(t)\, e^{-rt} \qquad 9$$

That is, Eqt.(9) identifies the objective of an economic agent. The investor wants to maximize the present value of the investment by making the optimal choice of *t*. To maximize present value, take the derivative of *PV*(*t*) w.r.t. *t* and set it equal to zero.

$$\frac{dPV(t)}{dt} = g'(t)\, e^{-rt} - rg(t)\, e^{-rt} = 0 \qquad 10$$

• In any optimization problem, the first order condition (FOC) identifies the value of the choice variable that optimizes the objective function. In this sense, eqt (10) can be solved for the value of *t* that maximizes present value. We call this value of *t*, *t**. Again, recognize the use of the star notation. Over all possible values of *t*, eqt.(10) picks out the one value that is optimal, that is, of couse, so long as the sufficient second order conditions (SSOC) are satisfied.

Most of the time, the FOC does not solve very explicitly for the value of the choice variable. Such is the case here. The solution is implicit. However, it is still revealing. Canceling terms in equation (10) results in the rule: $g'(t^*)/g(t^*) = r$. This says that the time to cut the tree is when its growth rate in percentage terms is equal to the interest rate. This the Golden Rule of resource economics and one you should never forget.

To clean up loose ends, the SSOC are defined as

$$\frac{d^2\, PV(t)}{dt^2} < 0 \qquad 11$$

The sign of this condition is determined by[4]

$$g''(t) - rg'(t) < 0 \qquad 12$$

which itself is satisfied if the second derivative of the growth function is negative, that is, if the growth function is concave. All this means that if the SSOC are satisfied, then the golden rule of harvesting, $g'(t^*)/g(t^*) = r$, is the optimal rule.

• Our model building process goes one step further, a very important step. When the FOC and SSOC are satisfied, *t* is transformed from a choice variable to a function. That is, *t**, the wealth maximizing value of *t* is really a function; it is a function of the parameters of the problem. In other words, our model of wealth maximization predicts behavior given any set of conditions or

[4] The second derivative of the objective function w.r.t. *t* is composed of two terms. One is the FOC times $-re^{-rt}$, which drops out because the FOC is equal to zero at the optimum.

constraints faced by the economic agent. The most important, and the most explicit constraint identified by our model is the interest rate, *r*; *t** is a function of *r*, i.e., *t**(*r*).

Obviously, to say that *t** is a function *t**(*r*), is not very enlightening unless we know something about the function. In fact, what we would like to know about the function is the sign of the derivative of *t** w.r.t. *r*. How does the optimal life of the tree change as the interest rate changes? Fortunately, even though the functional form of *t**(*r*) is not precisely identified, we can determine the sign of its derivative. We do this by differentiating the FOC w.r.t. *r* when the FOC is evaluated at *t**. That is, the FOC evaluated at the optimum is:

$$g'(t^*(r)) - rg(t^*(r)) = 0 = \frac{dPV(t^*(r))}{dt} \qquad 13$$

By the simple application of the chain rule, differentiating w.r.t. *r* gives:

$$\frac{d\left[\frac{dPV(t^*)}{dt}\right]}{dr} = \frac{d^2PV(t^*)}{dt^2}\frac{dt^*}{dr} + \frac{\partial^2 PV(t^*)}{\partial t\partial r} = 0 \qquad 14$$

or

$$\frac{dt^*}{dr} = -\frac{\left[\frac{\partial^2 PV(t^*)}{\partial t\partial r}\right]}{\left[\frac{d^2 PV(t^*)}{dt^2}\right]} \qquad 15$$

While the expressions in eqts. (14) and (15) may look obscure and almost daunting, differentiating the right-most side of eqt. (13) gives a meaningful solution:

$$g''(t^*(r))\frac{dt^*}{dr} - g(t^*(r))\frac{dr}{dr} - rg'(t^*(r))\frac{dt^*}{dr} = 0 \qquad 16$$

Rewriting gives:

$$\frac{dt^*}{dr} = \frac{g(t^*(r))}{g''(t^*(r)) - rg'(t^*(r))} \qquad 17$$

Notice that in both eqts. (15) and (17), the denominator on the right-hand side is the SSOC. Because of this we know the sign of this expression—as part of our model-building process we assumed the sign of the SSOC. In this single variable maximization problem, it is negative. And while the numerator of expression (15) is not obvious in general, in particular, as shown in eqt. (17), it is clearly positive.

Thus, we have derived a prediction from our model. We can unambiguously determine that the sign of the right-hand side of eqt. (17) is negative. Hence, the value of the derivative of optimal tree life w.r.t. the interest rate is negative. This is a comparative static result.

- It is instructive to visualize this problem. Pick a growth function, say t^b, where b is less than one. This gives a nice concave growth function starting from the origin. Now project each point on the growth path back to the vertical axis using the exponential function, e^{-rt}. The point where the exponential hits the vertical intercept is the present value of stopping the growth in the asset at that point on the growth function. (You can plot this using Excel or some such program. Let b be .5 and r=.1.)

Obviously, the problem is to maximize this present value. Graphically, we shift the exponential up to higher and higher points on the growth function. Project the exponential back to the vertical axis, but also project it forward. At low present values it cuts the growth function twice. However, eventually we are at a point where the exponential function is just tangent to the growth function. This is the maximum and it is characterized by our FOC: $g'(t^*)=rg(t^*)$.

Now think about what is happening in the picture you have drawn. Wealth is growth at a rate faster than the interest rate because it is held in the growth asset (trees, wine, etc.) up to t^*. However, after that point, wealth will grow faster by being held in the bank earning r. This is shown by the projection of the exponential function past t^*. At the maximum, the two margins are equal. The growth in wealth coming from the growing asset is g'. The opportunity cost is growth in wealth by taking the capitalized value g and putting in a security paying the return r in interest.[5]

- A simple but interesting extension of the Fisher Investment Problem is to allow the trees to be replanted immediately after they are harvested. The decision then is to determine when to cut each successive tree from now until the end of time. Since each tree is identical, we write:

$$PV = g(t)e^{-rt} + g(t)e^{-r2t} + g(t)e^{-r3t} + \ldots \qquad 18$$

The $g(t)$ expression stay the same for each tree; the discount factor increments to reflect the more distant nature of each future crop. Eqt. (18) is the sum of an infinite geometric progression, that is, the ratio of any consecutive terms is constant. From the *CRC Handbook*, $a_n=a_1\rho^{n-1}$, where a_n is the n*th* term, a_1 is the first term, and ρ is the ratio of consecutive terms. The sum to infinity is then equal to $a_1/(1-\rho)$. In the replanting problem e^{-rt} is the ratio term. This gives:

$$PV = \frac{g(t)e^{-rt}}{1-e^{-rt}} \qquad 19$$

[5] This makes sense. From the exponential function, the instantaneous rate of compounding is r. The derivative of with respect to t is r times e^{rt}. If we evaluate at t=0, the value of the derivative is r.

An alternative way of deriving the same result is to recognize that in eqt. (18) the sum of the terms starting with $2t$ and going up are equal to P times e^{-rt}. Making this substitution, collecting the P terms, and cross multiplying, gives eqt. (19).[6]

Differentiating PV w.r.t. t, setting the expression equal to zero, and simplifying "using some uninteresting algebra" gives

$$g'(\hat{t}) = rg(\hat{t}) + rPV(\hat{t}) \qquad 20$$

This formula says that the growth rate of each tree in dollars has to equal the forgone interest earnings that its capitalized (cut) value would earn plus the interest cost of the entire project from $\hat{t}$ on. The last expression on the right can be thought of as the rental cost of the land. Because rPV is positive, $g'(\hat{t})$ must be larger in the replanting case than in the case of the single tree. For $g'(\hat{t})$ to be larger, $\hat{t}$ must be smaller. That is, the trees are cut sooner.

However, the responsiveness of time life to the interest rate may be larger or smaller in the replanting case compared to the single project. The comparison of comparative static results is given by:

$$\frac{\partial \hat{t}}{\partial r} = \frac{g(\hat{t}) + PV(\hat{t})}{g''(\hat{t}) - rg'(\hat{t})} >< \frac{\partial t^*}{\partial r} = \frac{g(t^*)}{g''(t^*) - rg'(t^*)}.$$

While the expression look similar, they must be evaluated at different points along the g function. Hence, the rate of change in the optimal harvest time to the interest rate becomes conditional on the precise shape of the function.

■ THE INVERSE OF the investment problem is the Obsolescence Decision. Things wear out. There comes a time when it is optimal to scrap a machine. The point of this exercise is to identify the optimal shut down point.

Model the problem in the following fashion. Assume that a machine runs at a constant rate per period of time and produces revenues of v over each period. Let $R(t)$ describe the cost function for maintenance. This maintenance cost function is assumed to be positively sloped and convex. That is, maintenance costs increase at an increasing rate. Take a stylized example: A car last for ten years and is driven 10,000 miles per year. In the first year the only maintenance is oil changes. In the second, filters and other lubricants are replaced. In the third, a tune up is required. And so on, until in the tenth year, a new engine is needed and the vehicle is junked.

Formally, the problem is modeled by describing the present value function for the machine across all choices of machine life, t.

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[6] Substituting the value of the objective function for an expression on the right-hand side of the objective function is a trick that shows up quite commonly.

$$pv \;=\; \int_0^T [v - R(t)]e^{-rt}\,dt \qquad 21$$

This function is the integral of the revenue inflows and the maintenance cost outflows. The choice of T sets the terminal point, which is the upper limit of the integral.

For the optimal machine life, T^*, the present value of the machine's cash flows, $pv(T^*)$, is compared to the machine's cost, call this P. If $pv(T^*) > P$, the machine is profitable and is purchased. That is, profit can be written as $\pi = pv(T)-P$, but since P is not a function of machine life, max $\pi(T^*)$ is implied by max $pv(T^*)$, where T^* is found by differentiating $pv(T)$ and solving the first order condition (FOC) for T.

$$\frac{dpv}{dT} = [v - R(T)]e^{-rT} = 0 \qquad 22$$

The expression is simple: the derivative of an integral w.r.t. its upper limit is just the integrand evaluated at the upper limit. The expression further simplifies because e^{-rT} cancels out. The FOC says that the machine is scrapped when the maintenance cost is equal to the revenue flow, i.e., $R(T^*)=v$. In other words, when variable costs exceed revenues, shut down. Note that T^* in this problem is not a function of r or of P. Mechanically this is true because the cross partial of pv w.r.t. T and r or P is zero.

■ NOW CONSIDER THE more elaborate problem. Consider buying a machine that, if it is purchased, will be replaced when it wears out. That is, the investment project is not the acquisition of a single machine, but rather the decision to buy machine services from now on. Let the future be infinite. The capitalized value, V, of this project can be written as:

$$V \;=\; \int_0^T [v - R(t)]e^{-rt}\,dt - P + \int_T^{2T} [v - R(t - T)]e^{-rt}\,dt - Pe^{-rT} + \int_{2T}^{3T} [v - R(t - 2T)]e^{-rT}\,dt - Pe^{-r2T} + \ldots \qquad 23$$

Note the way that T enters the maintenance cost function. When the second machine comes on line at T its maintenance costs restarts at $R(0)$. Like the replanting problem analyzed before, this infinite series can be reduced by recognizing that the terms expressing the values of machines 2 and on can be expressed as the discounted value of the project.[7] Hence, the sum can be written as:

$$\max_{\{T\}} V \;=\; \frac{\int_0^T [v - R(t)]e^{-rt}\,dt - P}{1 - e^{-rT}} \qquad 24$$

Differentiating w.r.t. T, the FOC works out fairly simply to be

$$v \;-\; R(T^*) \;-\; rV(T^*) = 0 \qquad 25$$

The result says that if there are rents, i.e., positive value in the project, then the optimal life of each machine is shortened. If there are no rents in the venture then the optimal life is the same for the decision to retire one machine as for retiring a series of *n* machines. The result is actually a little more complicated. Machine life does not vary with respect to *v.* The change in *V* with respect to *v* is 1/*r*, which causes everything to cancel out. On the other hand, a change in *P* unambiguously decreases *V* and thus increases *T*. So the result is that as the price of the machine increases, the optimal life increases. If the price of the machine is too high relative to *v*, *V* becomes negative, and the project is abandoned.

The SSOC can be identified by differentiating eqt. (25) w.r.t. *T*.

$$-\frac{dR(T)}{dT} - r\frac{dV}{dT} = -R'(T) < 0 \qquad 26$$

The model allows us to ask and answer the comparative static question: How does machine life vary w.r.t. the interest rate? To find the answer, we differentiate the FOC w.r.t. *r*. This gives:

$$-\frac{\partial R(T^*)}{\partial T}\frac{dT^*}{dr} - V(T^*,r) - r\frac{\partial V(T^*,r)}{\partial r} = 0 \qquad 27$$

The last term on the left-hand side requires a moment of discussion. The derivative of *V* w.r.t. *r* follows the chain rule because *T** is a function of *r* and *V* is a function of *T**. The rationale is that *V* is a function of two arguments—the function *T** and the parameter *r*. Hence we write:

$$\frac{\partial V(t^*,r)}{\partial r} = \frac{\partial V}{\partial t}\frac{dt^*}{dr} + \frac{\partial V}{\partial r} \qquad 28$$

However, based on the envelope theorem, we know that the derivative of *V* w.r.t. *T* is zero, and we are left with only the partial of *V* w.r.t. *r* where *r* is not an argument in the optimal behavior function, *T**.

The partial of capitalized value, *V*, w.r.t. *r* is a fairly nasty derivative. Without laboring over it, let's just simplify as follows:

$$\frac{dT^*}{dr} = \frac{V\left[1+\frac{\partial V}{\partial r}\frac{r}{V}\right]}{-R'(T^*)} \qquad 29$$

The expression in the numerator is the elasticity of capitalized value w.r.t. the interest rate. This is negative because the interest rate is a cost. The denominator of eqt. (29) is negative by the SSOC. Thus, the derivative of machine life varies between positive and negative as the elasticity of *V* w.r.t. the interest rate varies between the elastic and inelastic ranges. That is, if *V*

[7] The expression can also be reduced by recognizing that it is an infinite geometric progression. It can be simplified by reference to the repeating terms.

is sensitive to the interest rate (the elasticity is greater than one in absolute value) then as the interest rate goes up, machine life goes up.

■ ASSUME THAT V are profits for a competitive firm and the market adjusts to drive profits to zero. That is, let v vary so that the present value described by equation (24) evaluated at the optimal T is just equal to P; call this v^*. This gives two equations. One (eqt. 25) describes the behavior of firms in choosing T. The other describes the behavior of the competitive equilibrium. The two equations are

$$\int_0^{T^*} [v^* - R(t)] e^{-rt} dt - P = 0 \qquad 30$$

$$v^* - R(T^*) = 0 \qquad 31$$

Now we can ask the question: How does behavior change as the interest rate changes? Note that this is a comparative static problem. There are two variables, T^* and v^*. These vary w.r.t. r. We differentiate the two equations w.r.t. r and solve simultaneously.

$$\begin{bmatrix} \int_0^{T^*} e^{-rt} dt & 0 \\ 1 & -R'(T^*) \end{bmatrix} \begin{bmatrix} dv^* / dr \\ dT^* / dr \end{bmatrix} = \begin{bmatrix} \int_0^{T^*} t[v^* - R(t)] e^{-rt} dt \\ 0 \end{bmatrix} \qquad 32$$

Solving by Cramer's rule, both dT^*/dr and dv^*/dr are positive. Actually, the equations are recursive. Since eqt. (31) is not a function of r, we can solve eqt. 30 directly for dv^*/dr, which is positive. Then we know that dT^*/dr is also positive because dT^*/dv is positive.

THIS SAME MODEL can be used to analyze the question of durability. Durability has been a topic of interest to economists for many years. The discussion usually focuses on the question of whether a monopolist will produce a good of greater or lesser durability. Sometimes the question is generalized by substituting the word quality for durability. The quality question is ambiguous; the durability question can be answered.

[Attached is a full manuscript. The following is a synopsis.]

Assume that the maintenance cost function has two arguments: time, t, and durability, α. Let the consumer optimize the function:

$$\max_{\{T,\alpha\}} V = \frac{\int_0^T [v - R(t,\alpha)] e^{-rt} dt - P(\alpha)}{1 - e^{-rT}} \qquad 33$$

where P is the price of the machine (car, washing machine, etc.) and it is a positive function of durability. R_α is negative; as durability increases, maintenance cost declines.

The FOC now become:

$$v - R(T,\alpha) - rV(T,\alpha) = 0 \quad 34$$

$$\int_0^T - R_\alpha(t,\alpha)]e^{-rt}\,dt - P_\alpha(\alpha) = 0 \quad 35$$

The second FOC says that the consumer chooses durability so that the cumulative marginal savings from durability over the life of the machine is equal to the marginal increase in price.

In the competitive market, the price of the machine is equal to its cost of production. Competitive firms sell consumers the level of durability that maximizes consumer value. Call this α^*. Notice that all consumers choose the same level of durability because optimal machine life is not a function of *v* (see the discussion in the context of eqt. 25, above).

It is useful to examine the distribution of buyers throughout the market. Figure 1 shows the model and depicts the population density to the left. Different people have different valuations of the service flow received from the machine. The competitive cost of manufacture will exclude some buyers from the market. As depicted, the marginal buyer in the competitive market is person *i*. Person *i* places a value of v_i on the service provided by the machine. Based on this and the cost of the machine, there is no consumer surplus for buyer *i*. This person is indifferent between buying the machine or not. Optimal machine life is found at T^* as determined by eqt. (34). Everyone else has a positive valuation.

Next consider what happens if the market is monopolized. The monopolist will increase the price of the machine. In this world the price of the machine will equal its cost plus the excess profit that the monopolist can extract. At a higher monopoly price some different buyer with a higher valuation, *v*, will become the marginal purchaser in the market. Let's suppose that it is person *j*. For this marginal buyer, the choice of the optimal life is unambiguously higher. The marginal buyer chooses *T* based on the rule given by eqt. (34). This is shown as T_0 in Figure 1.

Because the marginal buyer in the monopoly case operates the machine longer, the level of durability sold by competitive firms, i.e., α^*, is not optimal. Indeed, the monopolist will find it profitable to increase durability.

This can be seen in the following way. The rule that maximizes value in the competitive case is:

$$\int_0^{T^*} - R_\alpha(t,\alpha^*)]e^{-rt}\,dt = P_\alpha(\alpha^*)$$

However, in the monopoly case, consumers operate their machines longer. Hence,

$$\int_0^{T^0} - R_\alpha(t,\alpha^*)]e^{-rt}\,dt > \int_0^{T^*} - R_\alpha(t,\alpha^*)]e^{-rt}\,dt = P_\alpha(\alpha^*)$$

Thus, because the cumulative, discounted, marginal value of durability for consumers of the monopoly produce is higher (this is the left-most expression), the monopoly will increase the profitability of the machine by increasing durability.

More formally, the monopolist maximizes profit by maximizing the price it charges the marginal buyer while simultaneously excluding the revenue maximizing portion of the population. That is, profits are maximized by maximizing *P* times the cumulative distribution of the population that makes a purchase. Since $P=C+\pi$, the monopolist objective function in determining P becomes:

$$\max_{\{\alpha\}} \pi = \frac{\int_0^{T^+(\alpha)} [v_j - R(t,\alpha)]e^{-rt}dt - C(\alpha)}{1 - e^{-rT^+(\alpha)}}$$

where $T^+(\alpha)$ is the choice made by the marginal buyer, j, concerning the optimal life of the machine. The FOC of the monopolist is then:

$$\int_0^{T^+(\alpha^+)} - R_\alpha(t,\alpha^+)]e^{-rt}dt = C_\alpha(\alpha^+)$$

where $\alpha^+ > \alpha^*$.

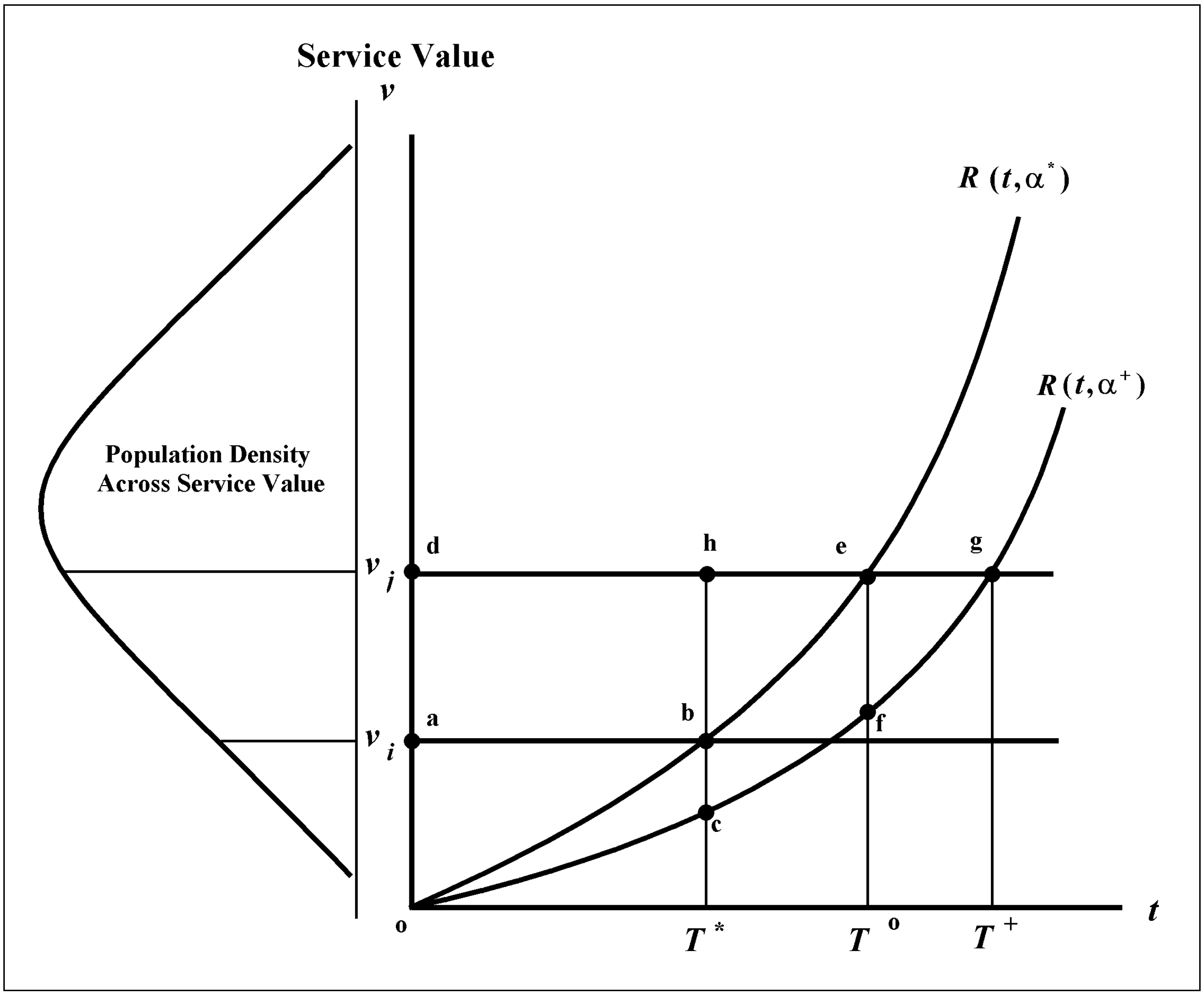

Figure 1

Taxation and Land Use[8]

ASSUME THERE ARE TWO PROJECTS of equal present value. That is, investors are indifferent between them. Project 1 generates immediate cash flows of $1 at each instant starting immediately and continuing forever. The other project, Project 2, has a gestation period of *T* that occurs before cash flows of $*c* begin, which then continue forever. Thus we have

$$P_1 = \int_0^\infty e^{-rt}\,dt = \frac{1}{r} \qquad 36$$

$$P_2 = \int_T^\infty c\,e^{-rt}\,dt = \frac{c\,e^{-rT}}{r} \qquad 37$$

and our assumption of equal present values gives

$$P_1 = P_2 = \frac{1}{r} = \frac{c\,e^{-rT}}{r} \qquad 38$$

or

$$c\,e^{-rT} = 1 \qquad 39$$

A graph of present values best illustrates the result. Notice that the present value of Project 2 is only equal to that of Project 1 at the beginning. As soon as Project 2 is commenced, its present value begins to grow at an increasing rate until it reaches a peak at time *T* where it flattens out at *c/r*. Also remember that the reason the two projects have the same present value at the initial point in time is that during the period 0 to *T*, the continuous cash flow of Project 1, which is $1, is foregone.

By way of example, if r is .1, then P_1 and P_2 are $10.

It is illustrative to graph eqt. (39). With dollars on the vertical axis and *T* on the horizontal, the l.h.s. of (39) is a downward sloping, convex function of the completion date, *T*, starting at the vertical intercept of *c*. We can think of the intersection of this function with a horizontal line at 1 as T^*. This says that the completion date T^* is the gestation period that equates the present values of the two projects. If Project 2 could commence before T^* then it would be preferred to Project 1. If Project 2 is delayed beyond T^* then Project 1 is favored. In this sense, T^* is an equilibrium value. It is the value of the gestation period that satisfies our assumption of equal present values. In the sense that it is an equilibrium value we use the *star* notation.

IMPOSE A VALUE TAX on Project 1. That is let there be a tax rate t that is levied against the present value of the project at each point in time. This is like a property tax levied on the assessed value of the land and buildings. The present value of Project 1 is now no longer P_1 because of the tax. Solving for the new present value, $\tilde{P}_1$, we have

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[8] Notes on Bentick JPE August '79

$$\tilde{P}_1 = \int_0^{\infty} (1 - \tau\, \tilde{P}_1)\, e^{-rt}\, dt = \frac{(1 - \tau\, \tilde{P}_1)}{r} \qquad 40$$

Notice that the tax liabilities are themselves a function of the present value of the project. Taxes are levied at the rate t on assessed valuation, that is, present value. To account for this mathematically, $\tilde{P}_1$ shows up on both sides of the equation but can be factored out directly to give:

$$\tilde{P}_1 = \frac{1}{(r + \tau)} \qquad 41$$

Again, by way of example, if τ is .02, then present value of project 1 falls to $8.33.

Now let us impose the same tax on Project 2 and recompute its present value now labeled $\tilde{P}_2$. Such a tax on Project 2 is slightly more complicated because of the fact that the present value of Project 2 is a function of time over the period 0 to *T*.

$$\tilde{P}_2 = \int_0^{T} [-\tau\, \tilde{P}_2(t)]\, e^{-rt}\ dt\ +\ \int_T^{\infty} [c - \tau\, \tilde{P}_2(T)]\, e^{-rt}\ dt \qquad 42$$

The first term on the r.h.s. is the present value of the taxes that must be paid during the gestation (or speculative) period where there is no postive cash flow to Project 2. These taxes rise continuously during the period 0 to *T* as the present value of Project 2 computed at each moment in time rises. Hence the expression $\tau\tilde{P}_2(t)$ represents these ever increasing tax payments. The present value of Project 2 reaches a maximum at *T*. The taxes paid from that time on are $\tau\tilde{P}_2(T)$. In characterizing $\tau\tilde{P}_2(t)$ and $\tau\tilde{P}_2(T)$ we can be more explicit. We know that $\tau\tilde{P}_2(t) = \tau\tilde{P}_2 e^{-rt}$ and $\tau\tilde{P}_2(T) = \tau\tilde{P}_2 e^{-rT}$ so we can write:

$$\tilde{P}_2 = \int_0^{T} [-\tau(\tilde{P}_2\, e^{rt})]\, e^{-rt}\ dt\ +\ \int_T^{\infty} [c - \tau(\tilde{P}_2\, e^{rT})]\, e^{-rt}\ dt \qquad 43$$

$$\tilde{P}_2 = -\tau\, \tilde{P}_2\, T\ +\ \frac{(c - \tau\, \tilde{P}_2\, e^{rT})\, e^{-rT}}{r} \qquad 44$$

$$r\, \tilde{P}_2 + r\tau T\, \tilde{P}_2 + \tau\, \tilde{P}_2 = c\, e^{-rT} \qquad 45$$

$$\tilde{P}_2 = \frac{c\, e^{-rT}}{(r + r\tau T + \tau)} = \frac{1}{(r + r\tau T + \tau)} \qquad 46$$

IT IS OBVIOUS, that $\tilde{P}_1$ is larger than $\tilde{P}_2$.

$$\tilde{P}_2 = \frac{1}{(r + r\tau T + \tau)} < \frac{1}{(r + \tau)} = \tilde{P}_1 \qquad 47$$

For instance, if *T* is 5, then the present value of project two in the event of a property tax is $7.69.

This means that the land tax has biased the choice of projects away from the one with the gestation period.

Again it is interesting to consider the problem in terms of the T that equalizes the two present values:

$$\tilde{P}_2 = \frac{c\,e^{-rT^*}}{(r + r\tau T^* + \tau)} = \frac{1}{(r+\tau)} = \tilde{P}_1 \qquad 48$$

Rewriting gives:

$$c\,e^{-rT^*} \;=\; 1 + \frac{r\tau}{r+\tau}T^* \qquad 49$$

This expression shows the effect of the property tax. Notice that expression (49) is very similar to eqt. (39). Let's graph (49) like we did eqt. (39). Think of $\{c\,e^{-rT}\}$ as a function of T. $\{ce^{-rT}\}$ is equal to 1 at the value T took before the tax was imposed. However, now the right-hand side of (49) is a positively sloping linear function of T starting at 1. The l.h.s., which is identical to the l.h.s. of eqt. (39), is a negatively sloping function starting at c. We see immediately that the new value of T that equalizes the present values of the two projects has gone down.

This conforms to our expectations. Because the value tax creates a bias against projects with long gestation periods, the effect is for investors to choose projects with shorter periods of gestation.

AS A FINAL NOTE, the effect of a tax on Project 2 that does not begin until Project 2 has positive cash flows has no biasing effect on the choice between Projects 1 and 2.

$$\hat{P}_2 = \int_T^\infty [c - \tau(\hat{P}_2 e^{rT})]e^{-rt}\,dt \qquad 50$$

$$= \frac{(c - \tau\,\hat{P}_2 e^{rT})\,e^{-rT}}{r} \qquad 51$$

$$= \frac{ce^{-rT}}{(r+\tau)} \qquad 52$$

$$= \tilde{P}_1 \qquad 53$$

This means, as Bentick points out, that a *tax on current use* such as an income tax or a property tax based on current production rather than speculative value leaves the choice of projects unaltered.

The policy implications of this analysis are somewhat profound. It used to be the case that property taxes were based on value assessments that occurred only when property changed hands. This created a kind of buffer between value in current used and speculative value. A piece of property could have a speculative value that was very high, but so long as it did not change hands, this value was not included in the tax bill. About ten years ago, a movement swept the country to begin continuous reassessment of property. Thus, farm land that had high speculative value but low cash flows was levied high taxes. The result predicted by Bentick's model is that this has caused more low value development. It is interesting to note that several states including South Carolina are now considering moving away from property taxation toward income and revenue taxes.

Question

Microtheory, September 12, 1997, #III:

Consumers buy appliances (e.g., washing machines) that last for a number of years. As the consumer keeps the appliance for more years, he spreads the initial cost of the appliance over more years. But he also incurs larger repair bills, because older appliances require more frequent repairs.

To answer the questions below, assume the following:

P = purchase price of an appliance,
w = price of an appliance repair,
n = total number of repairs over the useful life of the appliance,

where

$$n = \frac{u^{\beta}}{a}$$

and

u = useful life of the appliance ($u \geq 0$),
a = durability factor ($a > 0$), and
$\beta > 1$.

Assume throughout that

The interest rate is zero.
The scrap or trade-in value of the appliance is zero.
The price of a new appliance (i.e., P) is constant through time.
The energy usage for an appliance is the same, regardless of the age of the appliance.

a. What is the annual cost (or implicit rental price) for an appliance?

b. Explain how a consumer decides how long to keep the appliance. That is, determine the consumer's choice of the useful life (u) of an appliance.

c. Demonstrate that β is the elasticity of the total number of repairs over the useful life of the appliance with respect to its useful life.

d. Maytag (a manufacturer of appliances) advertises that its appliances need fewer repairs than appliances manufactured by other firms. Not surprisingly, Maytag appliances have higher purchase prices than appliances made by other firms. If Maytag appliances do need fewer repairs each year and do have higher prices than other appliances, would you expect Maytag appliances purchasers to hold their appliances longer than purchasers of other appliances? Explain. (Assume that β is the same for all appliance manufacturers.)

e. (Much harder--don't waste time here.) If Maytag appliances of a given age need, say, one-half as many repairs as any other manufacturer's appliances of the same age, by how much will the price of new Maytag appliances exceed the price of other new appliances?

Another question:
Compare the effect of property taxes on cars to sales taxes on cars.
Hint:
Applying the obsolescence analysis directly will give an answer in terms of the optimal life of the car. We can reasonably infer from changes in people's behavior in terms of the optimal life changes that they will make concerning the type of car they drive. That is, if one tax makes people drive their cars longer, then they will probably also buy more durable and nicer cars. The converse is also reasonably inferred.

Competition, Durability & Monopoly

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Competition, Durability & Monopoly

Abstract:
This paper shows is that when consumers choose the optimal life of the product, this choice is affected by price. The higher the price, the longer the product will be kept in service. If durability is a choice in the production process, more durability will be supplied as the optimal service life increases. This is true in both competitive and monopoly markets. However, the increased durability in the monopoly case is socially inefficient. Regardless, higher priced goods last longer.

INTRODUCTION

There has been much written on the durability issue and the discussion has been multifaceted.[9] One angle that has been pursued is the question of how a monopoly producer of a durable good can extract monopoly profits in one period when consumers recognize that the monopolist has an incentive to discount in future periods. Another piece of the durability puzzle involves suppression of durability: Suppose that GE invents a more durable light bulb that it can produce for exactly the same cost as the less durable ones. Will it produce the new durable bulbs or the old less durable ones or some of each? Another slant is quality.

Most of the results in this literature are narrowly linked to the specific conditions that frame a particular problem. This paper cannot claim to improve that situation, but it does draw some conclusions that seem to be empirically important. For instance, good stuff lasts longer than junk. We consider the special case of durability where product life is affected by the actions of both the manufacturer and the consumer, consumers are distributed normally in their demand intensity and uniformly across time in their repurchase decisions. The basic conclusion is that competition will provide more durability in a social optimal fashion as the cost of production increases. Durability is a substitute for more costly maintenance induced by the higher price. Similarly, monopoly pricing will result in more durability (when the good is sold) even though this durability increase is not socially optimal.

This result differs from the findings of other researchers (notably Schmalensee 1974, Su 1975, Rust 1986, and Waldman 1996) and is a result of the assumptions noted above. First we develop the model and then we discuss the nuances of the structure of our model that lead to different conclusions than those presented in the previous literature.

A MODEL OF OBSOLESCENCE

Things wear out. Many of the goods that people consume have the characteristic that they are great when they are new, but then as they get older, they are not as nice and they require upkeep. Repairs are costly. Eventually, the consumer scraps the old and re-buys. Cars and appliances are prototypical.

Let's start with a simple model. Assume that a machine runs at a constant rate and produces a constant use value of v at each point in time. Let $R(t)$ describe the cost function for maintenance also a continuous function of time. This maintenance or repair cost function is assumed to be positively sloped and convex. That is, maintenance costs increase at an increasing rate. Take a stylized example: A car last for ten years and is driven 10,000 miles per year. In the first year the only maintenance is oil changes. In the second, filters and other lubricants are

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[9] Carlton & Perloff review the literature. We give a thumbnail sketch later in the paper and show how our work fits in.

replaced. In the third, a tune up is required. And so on, until in the tenth year, a new engine is needed and the vehicle is junked.

Over one period, the value of the machine is given by the present value of the use value net of the maintenance cost. That is,

$$pv = \int_0^T [v - R(t)] e^{-rt}\, dt \qquad 1$$

where T is the life of the machine.

Consumption occurs over many periods. The consumer buys the machine, runs it for its optimal life, scraps it, and buys another. To capture this aspect of the problem, let P stand for the price of the machine, and let's assume that the consumer has an infinite horizon. Given this, the consumer's objective function looks like:

$$\max_{\{T\}} V = \frac{\int_0^T [v - R(t)] e^{-rt}\, dt - P}{1 - e^{-rT}} \qquad 2$$

where V is the net, discounted consumption value of the machine.[10] The consumer maximizes equation (2) by the choice of T. The consumer makes a choice of T once and then follows this same behavior period after period. The numerator of equation (2) is the single period value of consumption. The consumer spends P at the beginning of the period, $R(t)$ along the way, and gets the use value of v from the product at each moment. The denominator summarizes the process of replacement at the end of each period. The assumption that the consumer has an infinite horizon does not perfectly reflect any one consumer's choice set because individuals do not live forever, but it simplifies the math substantially without changing the substantive conclusions, and it is arguably the right way to model the problem in the case where an intermediate market leases machine services to individuals.

The first order condition is:

$$v - R(T^*) - rV(T^*) = 0 \qquad 3$$

which implicitly solves for the optimal product life T^*. There are a couple of things worth noting about equation (3). The optimal product life is affected by the net consumption value of the machine. Equation (3) says that the larger is the net consumption value, the more often the consumer turns over the product. Even so, the value of service flow, v, has no effect on machine life. This is because changing v in equation (3) has the primary effect of making the marginal repair costs look smaller and the secondary effect of increasing V. The effects wash. On the other hand, the effect of the price of the machine on optimal machine life is unambiguous. If the price of the machine increases, the consumer will increase machine life.

If there is no net consumption value, then equation (3) says that the consumer uses the machine until the cost of maintenance is exactly equal to the use value. The consumer then

[10] The setup of this problem is identical to the standard Fisherian growth model with replanting. It is very similar to the model used by Su (1975) and is adapted from Maloney and Brady (1988). This formula can be derived in several different ways. It comes from the general statement of the consumption process given by:

$$V = \int_0^T [v - R(t)] e^{-rt}\, dt - C + \int_T^{2T} [v - R(t-T)] e^{-rt}\, dt - Ce^{-rT} + \int_{2T}^{3T} [v - R(t-2T)] e^{-rT}\, dt - Ce^{-r2T} + \ldots$$

based on the assumption that the consumer has an infinite horizon.

scraps the machine and is indifferent between replacing it or engaging the next best alternative, which presumably determines *v*. For instance, if we are considering a washing machine, *v* can be thought of as the cost of taking one's clothes to the laundry, which would be the opportunity cost of doing them at home. Consumers who have no net consumption value from home laundry are indifferent about buying a new washing machine when the old one costs "too much" to fix.

THE CHOICE OF DURABILITY

Next let's consider what happens when the machines can be made more or less durable. To incorporate this into the model, let both *P* and *R* be functions of a durability factor, α. A more durable machine is one that requires less maintenance in order to yield the flow of services, *v*. This means that increased durability causes the maintenance cost function to shift down across the entire potential life of the machine. In other words, $\partial R(t,\alpha)/\partial\alpha$ is negative. This is the effect of durability from the consumer's perspective. From the manufacturer's side, in order to produce a machine that achieves these lower maintenance costs, the machine is necessarily more expensive. If *C* is the production cost of the machine, then $\partial C(\alpha)/\partial\alpha$ is positive and so too is $\partial P(\alpha)/\partial\alpha$ in a competitive market. We assume that the second derivative of cost with respect to α is also positive.

Incorporating the idea of durability as a choice variable, equation (2) becomes:

$$V = \frac{\int_0^T [v - R(t,\alpha)]e^{-rt}\,dt - P(\alpha)}{1 - e^{-rT}} \qquad 4$$

Consumer value is described by equation (4) and consumers choose *T* in order to maximize this value as described by equation (3). From the production side, competition forces firms to maximize equation (4) with respect to the choice of α. Competition yields the same result that would occur if the consumer integrated production and consumption of the machine.[11] Interaction in the competitive market place causes equation (4) to be maximized on the margins of both useful life, *T*, and durability, α. At zero profits, competitive firms are paid a price, *P*, for their machines that equals the cost of production given the optimal level of durability, $C(\alpha)$.

The first order condition of equation (4) with respect to α is:[12]

$$\int_0^T - R_a(t,\alpha)\,e^{-rt}\,dt = C_\alpha(\alpha) \qquad 5$$

This says that the competitive firms will adjust the durability of the machine until the marginal cost of durability, C_α , is equal to the discounted value of the maintenance cost savings generated by the additional durability. Competition causes equations (3) and (5) to be simultaneously solved at the market equilibrium. The optimized values, T^* and α^*, implicitly identified in equations (3) and (5) are found when these two equations are jointly satisfied.

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[11] Alternatively, we can think about the competitive market as one that integrates the choice of the optimal life and the optimal durability, and then leases machines to consumers. In this sense, the assumption of an infinite horizon is innocuous.

[12] R_α is the derivative of *R* with respect to α. C_α is similarly defined.

COMPARATIVE STATICS IN THE COMPETITIVE MARKET

It is useful to consider the characteristics of the competitive equilibrium. Of particular interest to us is the effect of an increase in cost on the behavior of the firms and consumers. Let's assume that a fixed cost associated with production increases such that the marginal relationship between cost and durability is unaffected. The same results obtain if we assume that the unit cost of production changes but the marginal cost of producing a more durable good remains constant. The point is to change cost without affecting the marginal cost function for durability.

The comparative static results of this experiment can be found from equations (3) and (5). Simply put, an increase in the fixed cost of production will cause the price of the product to increase, the optimal life of the product to increase, and the level of durability of the product to also increase. This last point is the most interesting. It is found by differentiating equations (3) and (5) with respect to a change in fixed cost, *F*. The result is:

$$\begin{bmatrix} -R_T(T) & -rR_\alpha(T) - rV_\alpha \\ -R_\alpha(T)e^{-rT} & \int_0^T -R_{\alpha\alpha}e^{-rt}\,dt - C_{\alpha\alpha} \end{bmatrix} \begin{bmatrix} \dfrac{\partial T^*}{\partial F} \\ \dfrac{\partial \alpha^*}{\partial F} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial P}{\partial F}\dfrac{-r}{(1-e^{-rT})} \\ 0 \end{bmatrix}$$

which based on the sufficient second order conditions yields the result that both optimal product life and durability respond positively to a change in cost.

It is enlightening to look at a graph. Consider a market made up of heterogeneous buyers who differ in the value *v* that they place on the service flows from the machine. Some people value the machine very little, some very much. Let this distribution of people across the service value follow the normal for illustrative purposes.[13] Figure 1 shows the model in the dimensions of *T* and *v*. The service value, *v*, is measured on the vertical axis. To the left of this is the density of the population of consumers as they vary in *v*. *R* is the repair cost function which is the same for all buyers.

In the competitive market, there are some consumers who do not value the machine enough to buy it. Furthermore, there are some consumers whose value of *v* is just sufficient so that the discounted value of *v* minus the discounted value of the maintenance expenditures is exactly equal to the competitive price of the machine. In Figure 1, this level is labeled v_i. For buyers with service value v_i, the area defined by the points *oab* discounted to the present is exactly equal to the price of the machine. All buyers with *v*'s higher than this enjoy consumer surplus from buying the machine.

Under the assumption of a competitive equilibrium, equation (5) holds and the marginal reduction in repair costs from additional durability is just equal to the cost of producing the additional durability. In terms of the graph, if competitive firms increased durability from α^* to α^+, the marginal benefit would be the reduction in repair costs which is given by the discounted value of *obc*. By construction, the marginal benefit would not cover the marginal cost of this improvement.

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[13] The shape of the distribution does not affect the conclusions. We assume that each consumer only operates one unit of the good at a time.

Now consider what happens if the fixed cost of production increases. The competitive price of the machine must increase.[14] Buyers at level v_i are priced out of the market. Some new buyers with a higher v become the marginal purchasers.

Let's assume for the moment that the market price rises to a value equal to the discounted value *ode*. Holding durability constant at α^*, this means that buyers with service value v_j are now marginal. Notice what happens as a consequence. The marginal buyers operate their machines longer. Given the higher competitive price, buyers use the machine until the marginal repair cost, $R(T)$, is equal to v_j. The optimal life of the machine is increased from T^* to T^{o}. The fact that consumers operate their machines longer as price goes up increases the marginal value of durability. Durability is a substitute for the higher maintenance costs they are incurring. As a consequence, competitive suppliers will produce a good of more durability.

This can also be seen in Figure 1. If the level of durability increases from α^* to α^+, the marginal benefit to consumers is the discounted value of *oef*. At the lower equilibrium price and shorter optimal life, the *obc* part of this area was marginally insufficient to cover the cost of the durability improvement. The cost of the durability improvement has not changed but its marginal benefit has increased by the amount *cbef*. Hence, competitive firms will now find it profitable to undertake this durability improvement. Because the consumer increases the life of the machine when the competitive price increases, the value of additional durability increases and competitive firms will supply it.

Of course, there is a feedback effect. The equilibrium is regained when the price of the machine, given the optimal durability, is equal to the discounted area under v and above R for the marginal buyer. Competition will drive price down to afford purchase of the machine to the largest possible number of buyers. For instance, $\{T^+,\alpha^+\}$ would define a competitive equilibrium if the discounted value of *odg* were equal to the cost of production and the marginal benefit of durability were equal to the marginal cost. The important result is that there is an unambiguously positive relation between the fixed cost of production and competitive equilibrium durability.

MONOPOLY

Next suppose that only one firm produces the machine. The monopolist is interested in maximizing the price of the machine above its production cost. Everything else the same, the monopolist would like to maximize V by the simultaneous choice of T and α given the production cost of the machine, $C(\alpha)$, and then charge a price for the machine that is equal to this maximized V. However, the monopolist cannot do this, at least not if it sells the machine. When the monopoly sells the machine, it looses control of T. The consumer makes the choice of the optimal life and the monopolist can only choose durability. Since the monopoly price is higher than the competitive price for the same cost of production, the marginal buyer in the monopoly market has a higher service value, v, and as a consequence a higher optimal life, T. This must be true since the marginal buyer in both monopoly and competition behaves according to:

$$v = R(T,\alpha) \qquad 6$$

and a higher v implies a higher T.

For instance, if competition yields an equilibrium characterized by $\{T^*, \alpha^*\}$ where the marginal buyers have service value of v_i, monopoly would increase price, drive more buyers out

[14] Depending on the nature of the production process, firms will likely exit the market as output price goes up and the number of units sold decreases.

of the market, and at the higher price, the new marginal buyers would choose a longer optimal life, say, T^+. The effect of monopoly pricing would be the same in this regard as an increase in the fixed cost of production in a competitive industry.

The model says that the effect on durability is the same also. Because the monopoly price causes consumers to increase the optimal life of the machine, the monopolist has an incentive to increase durability. In fact, the marginal durability condition for profit maximization for the monopolist is identical to the marginal condition for competitive firms.

The monopoly maximizes profit by choosing durability to maximize the difference between the price and cost of the machine, i.e., $\pi = P - C$. The consumer chooses the optimal life of the machine but the monopolist sets the price of the machine so that V in equation (4) equals zero for the marginal buyer. The monopolist picks the marginal buyer so that marginal net revenue across units sold equals zero. Let the number of units sold, Q, be:

$$Q = \int_{v}^{v_N} g(x)dx = 1 - G(v)$$

where $g(.)$ is the density of buyers, $G(.)$ is the cumulative distribution, v is the service value of the marginal buyer, and v_N is the maximum of service values across the population. Thus, the optimization function for the firm is:

$$\max_{\{\alpha,v\}} \pi = [1 - G(v)]\left[\frac{v}{r}(1 - e^{-rT^+}) - \int_0^{T^+} R(t,\alpha)e^{-rt}dt - C(\alpha)\right]$$

The partial of profits with respect to durability can be written as:[15]

$$\frac{\partial \pi}{\partial \alpha} = [1 - G(v)]\left[-\int_0^{T^+} R_\alpha(t,\alpha)e^{-rt}dt - C_\alpha(\alpha) + \frac{\partial \pi}{\partial T}\frac{\partial T^+}{\partial \alpha}\right] = 0$$

However, the partial of π with respect to T is the same as the partial of V with respect to T (i.e., equation 6) which is equal to zero. Thus, we have:

$$-\int_0^{T^+} R_\alpha(t,\alpha)e^{-rt}dt = C_\alpha(\alpha) \qquad 7$$

Equation (7) for the monopolist is identical in form to equation (5) for the competitive industry.[16] However, given the same cost of production, the optimal life chosen by consumers in the monopoly case is greater than that chosen under competition; i.e., $T^+ > T^*$.

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[15] The first order condition with respect to the number of units to sell is: $\frac{\partial \pi}{\partial v} = Q\frac{\partial (P-C)}{\partial v} + [P-C]\frac{\partial Q}{\partial v} = 0$, which says that the firm sets marginal net revenue equal to zero. This expression is equal to $\frac{1-G}{g} = \frac{r}{1-e^{-rT}}(P-C)$.

[16] This is the point on which our model differs from Su (1975). We discuss this in the next section.

What happens in monopoly is that consumers run their machines longer to avoid the higher price charged by the monopolist. Because the monopolist knows they will run the machines longer and consequently incur higher maintenance costs, the monopolist lowers the maintenance expenditures by making the machines more durable. In this way the monopolist can charge a higher price for the machine. Essentially, in recognizing that the consumer will spend more on maintenance as the price of the machine goes up, the monopolist sells the consumer cheaper maintenance by increasing durability.

The response of the monopolist is identical to the response of the competitive industry. However, it is an inefficient response in the monopoly case. Let's assume that α^+ is the profit maximizing level of durability for the monopolist and that given the same cost function for production the competitive equilibrium would be given by α^*. From Figure 1 we see that in the competitive market, buyers with service value v_j would enjoy consumer surplus equal to the discounted value of *odhb*. However, the monopolist is not able to extract this much. The monopolist can only charge a price equal to the discounted value of *odg*, which is less (because it repeats less often). The monopoly price drives an artificial wedge between price and cost, albeit one that creates a behavioral response among consumes that is just like a true, competitive cost increase. The inefficiency results because consumers spend too much on maintenance relative to the true cost of production.

This result is consistent with the standard conclusion in the literature that the monopolist makes more profit renting rather than selling. The monopolist can increase profits if it is able to stop the consumer from making wasteful expenditures on maintenance. For instance, if the firm can monopolize both the manufacturing and service, it will force consumers to turn over the machines more rapidly and thereby extract more surplus. This point is discussed more later.

It has been suggested by some readers that the monopolist would like to make the machine wear out sooner in order to counter act the incentive of the consumer to stretch out product life. There is no doubt that the monopolist wishes it could make consumers purchase the product more frequently. Even so, the *ceteris paribus* comparison between monopoly and competition shown in Figure 1 holds under the assumption that the repair cost function *R*(.) is the same for competitive and monopoly firms.

EXTENSIONS & APPLICATIONS

The model is developed based on maintenance as the factor that affects the consumer's choice of optimal life. However, the result is more general. Any time a product exhibits declining value as it grows old, there is an optimal turnover/durability relationship. And, in such cases, higher prices will induce consumers to hang on to old products longer, thereby inducing manufacturers to supply more durability.

In the general framework of the problem, the consumer objective function is:

$$V = \frac{\int_0^T N(t,\alpha)\, e^{-rt}\, dt - P(\alpha)}{1 - e^{-rT}} \qquad 8$$

where *N*(*t*) represents the net value of service flows to the consumer and is a declining function of time. The consumer chooses the product life *T* while the firm chooses α. The consumer is induced toward longer product life as product price increases. The firm's choice of α equates the marginal effect of α on consumer value with the marginal cost of α. As the consumer chooses longer product life, the marginal value of durability increases and the firm supplies more of it.

The prediction of the model is the same whether the price increase faced by consumers is the result of a true competitive cost differential or a monopoly markup. In the monopoly case, if there is a way to force consumers to adopt the optimal life, the monopolist does it. However, if consumers have the ability to stretch out the life, then the monopolist is induced to make the product last longer.

Consider the case of lipstick. Lipstick wears out. As it gets older, it loses both its suppleness and the brightness of its color. The competitive equilibrium in lipstick is for a unit to be worn out in pliability and color when it is used up. However, when lipstick is priced to include a brand name premium, consumers will use less, saving it for special occasions. As a consequence, the high price firm (monopolist?) recognizing this incentive on the part of consumers will make its lipstick last longer so it can charge more for it. The prediction is that generic brands of lipstick dry out quicker than national brands.

The same is true for mattresses. The comfort afforded by mattresses deteriorates over time. Brands of mattresses differ in price. The prediction of our model is that high priced mattresses will be more durable than cheaper ones. Arguably when a mattress is made more comfortable it may lower the marginal cost of making it more durable. However, over and above any marginal cost justification, firms that charge a higher price either because of monopoly power or brand name quasi-rents will be induced to increase durability.

Other Models

Our model reaches conclusions that are not part of the current literature on the durability problem. We will try to reconcile the apparent conflicts. There have been numerous papers written on the various aspects of the durability problem. These models can be usefully organized by the way they characterize the period of production and consumption. At one extreme, we have the problem that has come to be called the Coase Conjecture. Coase (1972) introduced the idea. Bulow (1982) formalized the model in a two-period setting. There have been many contributors since. In the problem posed by Coase, all buyers enter the market together each period. The good that they buy lasts forever. The issue is when will they buy it and at what price. The conclusion, as stated by Carlton and Perloff, is that the monopolist in such a setting is limited in its ability to extract consumer surplus by the fact that consumers rationally expect that the good will be discounted in the future.

At the other extreme, we have the classic light bulb problem. In the light bulb problem, consumers are distributed uniformly (or randomly) across time. The good that they buy wears out and they must revisit the market periodically. In this case Swan (1970) and Sieper and Swan (1973) show that monopoly and competition supply the same level of durability.

An important difference between these two durable good problems is the time distribution of consumer demand. When consumers must revisit the market periodically, the price discounting that follows from the Coase Conjecture is not at play. If each period is composed of some consumers who are replacing burned out light bulbs and other consumers who have bulbs still burning bright but who would buy more if the price fell, there is no incentive for price cutting. From the seller's perspective the period of production repeats indefinitely and to lower price in the second period of consumption for some consumers would be to lower price in the first period for others. Obviously, such a strategy would be sub-optimal.

The consumption setting analyzed in the this paper focuses on those markets where consumers must repurchase periodically and where they reenter the market randomly across time. Thus, the issues raises by the Coase Conjecture can be ignored. For the goods that we consider, all consumers operate their machines for the same, optimal length of time. However, the start and finish dates are different across consumers. The manufacturer, competitive or monopoly, sees a constant flow of demand through time that is a function of market price.[17]

The durability problem that we consider is different from the light bulb problem in that consumers choose the life of the product. This extension was developed earlier by Schmalensee (1974) and Su (1975) in much the same way as done here. The result of this prior research is that the monopolist in such a setting will sell a product with an inefficient amount of durability. However, neither model is more specific. As Schmalensee (1979) says in a review article, "No simple condition indicates whether built-in durability will be higher or lower [in monopoly] than under competition." The model of Su is nearly identical to ours so it is surprising that he does not reach the ultimate conclusion that we do. However, he does not explicitly consider the value of service or the distribution of buyers across service value. Rather, his model is built on minimizing the average cost of consumption of the durable good by a representative consumer. As a consequence, he gets an ambiguous result.

Another question that is raised in the literature on durability notably by Rust (1986) and Waldman (1996) is the effect of the secondary or used market for the good. Both of these papers find that the monopolist under produces durability because of the used market. However, these results are not in conflict with ours because we look at markets that have no used goods. It is assumed here that while consumers differ in terms of the value that they place on the service flow from a machine, they all choose the same optimal life, which of course is dependent on the price of the good. Hence, they all consume the good until it is scrap. This means that there is no second-hand market and none of the associated pricing problems.

The contrast in the consumption decision between text books and home appliances highlights this distinction. The textbook market (for the most part) is characterized by consumers who use the good for one period and then have no further use. Thus, after their period of consumption, the good returns to marketplace. On the other hand, for many durable goods there is little or no market for used goods. When mattresses are done, they are pitched. There is only a limited market for used household appliances. Few homeowners go to the Goodwill store to shop for dishwashers. However, they do make a choice between having their old dishwasher repaired or buying a new one. Hence, there does seem to be a number of durable goods where the used market is not at play.[18]

But what about cars? In the automobile market there is an active re-circulation of used vehicles. Indeed, for some models, the label is "pre-owned" rather than used, presumably

[17] Of course, in practice, there are markets that have both characteristics. Take, for instance, computers. Consumers enter and reenter the computer market. All consumers enter the market together, though some are excluded by price, every time a new technology is introduced. However, consumers reenter the market randomly as their units break down. We ignore these markets as well.

[18] However, there is one intriguing market where resale seems to be an issue. It is the market for musical instruments. The period of consumption of musical instruments may be more like textbooks than household appliances. What happens is that people buy a musical instrument, often for their child. After some period, the instrument either becomes a part of the long term consumption habit of the family and is kept forever or it is not used and returns to the market in the form of a used good. The intriguing thing is that there are two major Japanese manufacturers of pianos. One has a warranty policy that actively discourages the used market, while the other is more supportive. This pattern seems to follow the model of Waldman (1996).

indicating that they are just as good as new. A couple of points are worth noting. First, there seems to be less substitution among buyers in the new and used car market than in the textbook market. People who buy new cars tend to be separate from those who buy used cars. New car purchasers are on one consumption cycle and used car buyers are on another. This suggests that the goods are different in some ways to some buyers even though they are similar in other ways. Second, if the new and used market is composed of different buyers, then the manufacturer is not competing against itself in its sales of new vehicles. Hence, the model developed here arguably applies even to the market for automobiles.

Antitrust Cases

There have been at least two antitrust cases that involve maintenance and to which the theory developed here adds some insight. The first is *United Shoe*.[19] United Shoe controlled between 75 and 85 percent of the shoe machinery market in the 1953. The company had a policy of only leasing its machines and not selling them. It also provided free maintenance. In the context of our model, the lease arrangement is explained in terms of enforcement of optimal life. If the company was charging a monopoly price for its machines, then users would have the incentive to run the machines too long. By our argument, United Shoe chose to lease as a way of forcing a profit maximizing turnover of machines. United Shoe made the decision about when a machine required replacement by the structure of its lease. Posner and Easterbrook remark on the leasing policy of United Shoe: "Perhaps the leasing policy gave shoe makers an incentive to place orders for new machinery at regular … times."

The United Shoe case is often given as an example of the Coase Conjecture. Based on the conjecture, firms extract more profit from renting rather than selling because if they sell, they cannot credibly commit to not reduce price in future periods. The argument presented here is different. Here we argue that the firm would rather rent so that it can optimally determine the durability of the machine. If it sells the machine, it knows that the consumer will run the machine longer and it knows that it will make the machine too durable as a consequence. It can recapture some of that dead weight loss by renting the machine, making it less durable, and forcing replacement when optimal. This prediction is independent of the Coase Conjecture.

The issue is revisited in *Eastman Kodak Co. v. Image Technical Services, Inc., et al*.[20] Kodak was sued by independent service organizations because it attempted to restrict the secondary market on the service of its copying machines. The court in a divided opinion ruled against Kodak. The issue felt by the court involves the *per se* nature of the precedents on tying. Both sides of the court seemed willing to admit that Kodak had no market power in copying machines. The explanation for attempting to limit the secondary market on service was well developed by Kodak. The court's main argument in favor of prohibiting the limitations was that Kodak was able to exercise market power over customers who were locked into the product for the short run after they purchased the machine and did not anticipate the maintenance cost. Kodak claimed that customers buy the machine and maintenance as one service.

The majority's opinion in the Kodak case is not compelling. It is hard to believe that there are significant profits, especially over the long haul from beating locked-in customers out of a few dollars on maintenance that they did not expect. A more reasonable argument is that

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[19] United States v. United Shoe Machinery Corp., District Court, D. Massachusetts, 1953, 110 F.Supp. 295; affirmed 347 U.S. 521 (1953). See discussion in *Antitrust Cases, Economic Notes and Other Materials*, Richard A. Posner and Frank H. Easterbrook, West Publishing, 1981; durable goods, pp 624-627; United Shoe, pp. 628-643.

[20] 112 S.Ct. 2072 (1992).

Kodak had a niche in the market for copiers and sought to extract the full measure of its albeit not large market power. For consumers who would otherwise enjoy a surplus from using Kodak copiers over the market alternative, Kodak's monopoly price would cause these consumers to spend too much on maintenance and not turn the copiers over at the optimal time. By monopolizing service, Kodak was able to price maintenance in a way that would counter this incentive. In the context of Figure 1, by monopolizing service Kodak shifts up the *R*(.) function at the same time that it charges a monopoly price. From an efficiency perspective, allowing Kodak to control the price of service, minimizes the amount of wasteful maintenance. This is true in spite of the fact that Kodak captures all of this extra surplus.

CONCLUSION

Durability is a complicated problem, especially in the many cases where durability is affected by the behavior choices of both consumers and producers. This paper shows is that when consumers choose the optimal life of the product, higher product price will induce a longer product life, which causes firms to respond by producing a more durable product because the marginal value of durability is increased.

This response of firms is the same for competitors and for monopolists. In the competitive case the increased durability is efficient. In the monopoly case, the increased durability is not, which is why the monopoly firm would rather rent than sell. When monopolists sell the product, they usually lose control of the repurchase decision. Possibly they can retain control by monopolizing service. However, an alternative is not to sell the product in the first place, but rather to lease it and force the profit-maximizing turnover rate on consumers.

At all events, higher priced goods are predictably more durable than lower priced goods, and this prediction is invariant with respect to market structure. Hence, in the case of brand name products, generic brands will be observed to wear out more rapidly than nationally branded products.

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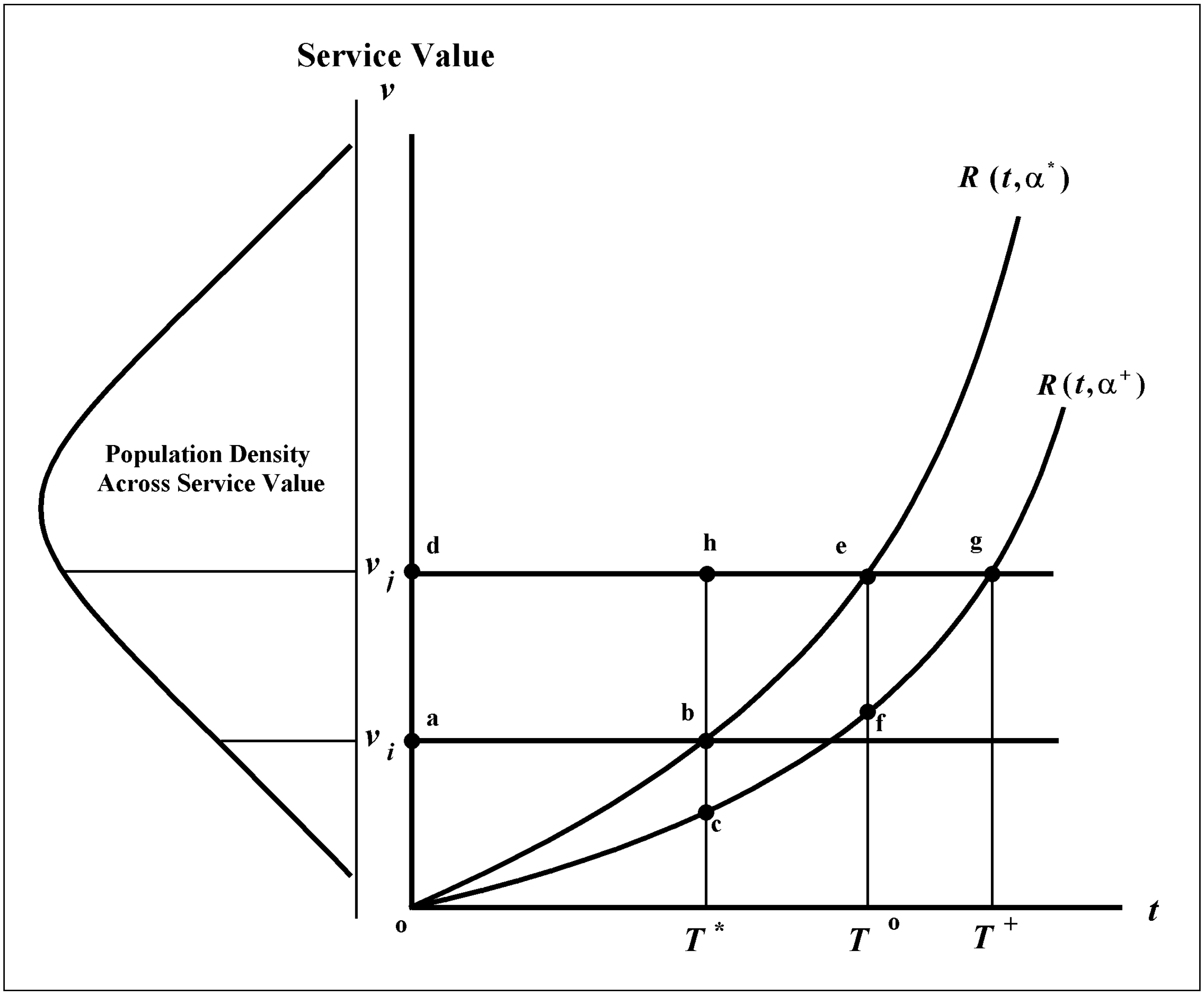

Figure 1