

## ■ UNCONSTRAINED OPTIMIZATION<sup>1</sup>

The problem of the **Market Separating Monopolist** is conceptually simple and yet technically difficult. For that reason it is a problem that advanced price theory students should be able to handle and should be excited to expound upon. I leave it to you to grade your own resolve.

- The Model is based on a firm that produces one product but sells it in two different markets. The monopolist separates the markets so that units sold in market 1 do not flow back into market 2. In this way the monopolist is able to charge two different prices for the same product. Moreover, the monopolist finds that it is almost always more profitable to charge different prices than to charge the same price.

Intuitively, we can think of the monopolist's problem in 2 parts. The first part is one of allocating a fixed supply of its product between the two markets. The monopolist will shift supplies between the two markets until the marginal value of a unit is the same in both markets. If this were not the case, then a profit opportunity would exist. The marginal value of one market compared to the other is the *marginal revenue* in one market compared to the other, again assuming that the total supply is fixed. Profit maximization requires an equalization of these two margins.

Next the monopolist has to decide how many units to produce in total. This does not really complicate the problem because the monopolist need only compare the marginal value of a addition unit *in either market*, since they are identical, to the marginal cost of producing an additional unit.

- The mathematics of this problem are simple and are almost too terse even though they reveal precisely the intuitive rule outlined above. The firm maximizes profits across the two products:

$$\max \Pi = R_1(q_1) + R_2(q_2) - C(q_1 + q_2) \quad 1$$

Notice that the cost function is defined in terms of total output, which is the sum of the output allocated to each market. Again this is consistent with the intuitive approach—the firm produces a batch of output at cost  $C$  and then ships it off to two different places.

The FOC are:

$$\Pi_i = MR_i - MC = 0, \quad i = \{1, 2\} \quad 2$$

These say that marginal revenue in the first market must equal the marginal cost of the last unit produced *across both markets*. Marginal cost is marginal cost at the last unit produced in total. Marginal revenue in the first market is the change in revenue that results from sending the last unit to market 1. For instance, if Michael Jackson is giving concerts in Bangkok and in Tokyo, the marginal revenue in Bangkok is determined by the number shows presented there while the marginal cost is determined by the total number of shows given on the tour.<sup>2</sup> Eq. (2) says,

<sup>1</sup> Silberberg (1990),(2001) sections 4.4, 6.2, 7.2, 7.3

<sup>2</sup> This example was suggested some years ago when Jackson was in the news in America.

however, that the marginal revenue in market 2 must equal marginal cost as well, and from this we get the intuitive result that marginal revenue in market 1 must equal marginal revenue in market 2.

- The graph of this problem becomes tricky and immediately reveals the extent to which the student is tracking on the technical aspects of the problem. Construct the graph as follows: draw two separate demand curves. Don't let them intersect. Next consider the marginal revenue curves associated with each. Add the marginal revenue curves horizontally. The horizontally summed marginal revenue curve identifies the total output requirement necessary to supply both markets when the two markets yield identical marginal contributions to profits relative to each other. The point where marginal cost cuts the horizontally summed marginal revenue identifies the profit maximizing level of total output. This total output is then distributed back to each market and the price in each market is determined along the demand curve in each market based on the output sent there.
- The comparative statics hinge critically on the SSOC in this problem, so let's investigate them thoroughly. They are:

$$MR_i' - MC' < 0, i = 1, 2;$$

$$\begin{vmatrix} MR_1' - MC' & -MC' \\ -MC' & MR_2' - MC' \end{vmatrix} > 0 \quad 3$$

What are the various possibilities associated with the satisfaction of these conditions. First, the conditions do not impose any explicit sign restrictions on marginal cost and revenues. However, there are a number of restrictions that we can deduce:

- If marginal cost is negatively sloped, then both marginal revenues must be negatively sloped.
  - If marginal cost is positively sloped, then one but not both marginal revenues can be positively sloped, and the sum of the slopes of the marginal revenues must be negative.
- You should be able to prove these propositions.

The implication of these restrictions come home relatively quickly. If, for instance, we consider the effect of an output tax, the model looks like this:

$$\max \Pi = R_1(q_1) + R_2(q_2) - C(q_1 + q_2) - t \cdot (q_1 + q_2)$$

The FOC are hardly changed:

$$\Pi_i = MR_i - MC - t = 0, \quad i = \{1, 2\}$$

and the SSOC are not changed at all. The comparative static results are found by differentiating the FOC at their solution values w.r.t.  $t$ . This gives:

$$\begin{aligned}
 MR_1'(q_1^*(t)) \frac{dq_1^*}{dt} - MC'(q_1^* + q_2^*) \frac{dq_1^*}{dt} - MC'(q_1^* + q_2^*) \frac{dq_2^*}{dt} - 1 &= 0 \\
 -MC'(q_1^* + q_2^*) \frac{dq_1^*}{dt} + MR_2'(q_2^*(t)) \frac{dq_2^*}{dt} - MC'(q_1^* + q_2^*) \frac{dq_2^*}{dt} - 1 &= 0
 \end{aligned}$$

This simplifies to:

$$\begin{bmatrix} MR_1' - MC' & -MC' \\ -MC' & MR_2' - MC' \end{bmatrix} \begin{bmatrix} \frac{dq_1^*}{dt} \\ \frac{dq_2^*}{dt} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

which is our familiar comparative static structure. The matrix on the left is made up of the elements of the hessian identifying the SSOC. The vector on the right side of the equation is the derivative of the FOC where the parameter,  $t$ , stands alone. Solving by Cramer's rule gives:

$$\frac{dq_1^*}{dt} = \frac{MR_2'}{H} \text{ and } \frac{dq_2^*}{dt} = \frac{MR_1'}{H} \quad 4$$

While neither of these expressions can be signed explicitly, we know that at least one, if not both, must be negative, and that the sum of the change in output must be negative. (You should be able to prove these propositions as well.)

Equation (4) tells us about quantity changes. This information can be used to infer something about price changes. For instance, how does the price difference between the two goods change as the tax changes? This can be answered as follows:

$$\frac{\partial(P_1 - P_2)}{\partial t} = \frac{\partial P_1}{\partial t} - \frac{\partial P_2}{\partial t} = \frac{\partial P_1}{\partial q_1} \frac{\partial q_1^*}{\partial t} - \frac{\partial P_2}{\partial q_2} \frac{\partial q_2^*}{\partial t} \quad 5$$

The change in price with respect to a change in the tax is given by the change in price wrt a change in quantity times the change in quantity with respect to the change in the tax.

Substituting for the change in quantity, we have:

$$\frac{\partial(P_1 - P_2)}{\partial t} = P_1' \frac{MR_2'}{H} - P_2' \frac{MR_1'}{H} \quad 6$$

where  $P_i'$  is the slope of demand in the  $i$ th market.

From this we can write:

$$\frac{\partial(P_1 - P_2)}{\partial t} >< 0 \text{ as } \frac{MR_2'}{P_2'} - \frac{MR_1'}{P_1'} >< 0 \quad 7$$

Notice that for linear demand curves the ratio of the slope of marginal revenue to the slope of demand is always constant, so the difference in price between the two markets is unchanged by a change in the tax rate.

We also might be interested in the price ratio.

$$\frac{\partial\left(\frac{P_1}{P_2}\right)}{\partial t} = \frac{\frac{\partial P_1}{\partial t} P_2 - \frac{\partial P_2}{\partial t} P_1}{P_2^2} \quad 8$$

and we can substitute from eqt. (4) above:

$$\frac{\partial P_i}{\partial t} = \frac{\partial P_i}{\partial q_i} \frac{q_i^*}{\partial t} \quad 9$$

From this we can write:

$$\frac{\partial\left(\frac{P_1}{P_2}\right)}{\partial t} = \frac{P_2 P_1' MR_2' - P_1 P_2' MR_1'}{P_2^2 H}$$

From this we see that:

$$\frac{\partial\left(\frac{P_1}{P_2}\right)}{\partial t} >< 0 \text{ as } P_2 P_1' MR_2' - P_1 P_2' MR_1' >< 0 \quad 10$$

or

$$\frac{\partial\left(\frac{P_1}{P_2}\right)}{\partial t} < 0 \text{ as } \frac{P_1}{P_2} > \frac{\left(\frac{P_1'}{MR_1'}\right)}{\left(\frac{P_2'}{MR_2'}\right)} \quad 11$$

Assume that  $P_1$  is the higher price and demand is linear. Then the price ratio between markets one and two will be greater than one. For linear demand the ratio of the slope of demand to the slope of marginal revenue is constant and the ratio of these ratios will equal one. Hence, eqt. (11) says that the price ratio between the markets goes down as the tax increases.

Other possible models: Assume that a tax is levied on output in only one market.

- Will this have any effect on output in the second market?
- If so, is it possible for output in the second market to increase?

Assume that the taxing authorities in market one levy a production tax but also mandate that output cannot decline. (They do this quite commonly by mandating that service be provided to all demanders and by capping the price of service at the so-called pre-regulation level.)

- Will this have any effect on output in the second market?

**Third Degree Price Discrimination**

