Optimal Control Theory[1]

Consider a resource *z* that is used in a productive process that yields output, *q*. Let the resource be an input flow taken from a stock. An example would be a fish catch taken from a fish population. Label the inverse of the stock of the resource *D*. Thus, the catch is *z*; the fish population is 1/*D*; and the amount of fish put on the table is *q*. Another example is ground water. Water pumped out of the ground is *z*. The depth of the water table is *D*.

The interesting aspect of the problem is that the resource regenerates itself at some natural rate. Call this *R*. The regeneration rate is assumed to be constant. Because of the natural regeneration rate, use of the resource is stable in the long run when $z = R$. This means that the amount of the resource taken away by use is exactly replaced by the natural regeneration. In the fishery example, the catch is replaced by repopulation by the remaining fish. When $z = R$, the stock of fish is constant. If $z > R$, outflow exceeds replacement and the stock of the resource falls. The dynamics of the stock can be characterized by

$$\frac{\partial D}{\partial t} = \dot{D} = f(z - R)$$

where $f' > 0$. Note that $\dot{D} = f(0) = 0$ and that the derivative of *f* is positive at 0 as well.[2]

The economy is organized around consumers whose utility is given by

$$U = U(q(z)) + x$$

where *U* is the valuation of *z* through the output *q*, and *x* is leisure or some composite commodity of goods that are not produced by use of *z*. Consumers are endowed with an amount of *x* each period. Label this $\bar{x}$.

On the production side of the economy, firms exploit *z* by applying resource to capture it. Let there be two resources *e* and *k*. The capture function is

$$z = z\left(\frac{e}{D}, \frac{k}{D}\right)$$

which embodies the idea that the stock of the resource affects the inputs necessary to exploit it. Let the relation between the resource stock and the input usage be linearly homogeneous. That is, if the stock falls by half, it takes twice as much *e* and *k* to obtain the same amount of *z*.

Now let's simplify the model by assuming that each person is also a firm producing *q*. This just suppresses explicit consideration of the *q* market. People have a budget constraint based on their per period endowment of *x*. From $\bar{x}$ consumers can consume *x* and buy resources *e* and *k* that are transformed into *z* , which is *q*, which is happiness *U*. The consumer *qua* firm minimizes

[1] This lecture is taken from Bob Deacon, "Incomplete Ownership, Rent Dissipation, and the Return to Related Investments," *Economic Inquiry* October 1994. See also, Silberberg (1990) Ch 18. Nicholson (1998) Ch 23, p 708-714.

[2] Lastly, there is a minimum *D*. That is, $D \geq \delta$. This means that to exploit the resource, there is a minimum depth that must be drilled, or the fish stock is not infinite, etc. We are not going to concern ourselves with this aspect of the problem.

the cost of producing *z* by choosing the optimal amounts of the two inputs. These inputs have constant prices p_e and p_k. In other words, the firm's behavior is

$$\min_{\{e,k\}} C = p_e e + p_k k, \text{ subject to } z(\cdot) = z_0$$

This minimization objective yields a cost function with the input prices, *z*, and *D* as arguments.

$$C^* = h(p_e, p_k, z)D$$

C^* is linearly homogeneous in *D* and convex in *z*, i.e., C_z, C_{zz}, and $C_{zD} > 0$.[3]

The equilibrium in the consumer and producer market for output is summarized by the consumer's objective function rewritten in the following fashion:

$$W = U(z) + x - h(z)D$$

Note that at the firm level *D* is a constant. However, at the market level aggregate exploitation of *z* affects *D*. As discussed above the steady state in *D* is defined by $z = R$. The states of the world vary in *D*.

Consider the following examples:

	Fisheries:	Groundwater:	Water Treatment:
z	catch	water	clean water
q	food	crop harvest	drinking
D	1/fish population	depth	dirtiness
R	repopulation	recharge	natural purification
e	fuel	fuel	labor
k	capital	pump	capital

The problem is one of dynamic optimization. The decision variable at the firm level is to choose *z* at each instant in time. Depending on the property rights regime, the consumer/firm may or may not be forward looking.[4] That is, *z* is a time variable *z*(*t*). *D* also follows a time path, so we write *D*(*t*). At t_0, *D* has an initial value, D^0. The function *f* (.) describes the path of *D* through time. Also, there is the constraint that *D* maintains a minimum level; we will ignore this complication. The objective is to maximize the present value of welfare by the choice of the optimal path of *z*. This is written as:

[3] Deacon is interested in developing the case of two inputs and the effect on optimal input combinations under alternative property rights regimes. If the model is simplified to one input, the homogeneity of cost w.r.t. *D* is seen immediately.

[4] In a world of common access property, into which water use usually falls, firms do not look past the current time period because there is no incentive to save for tomorrow. In a world of well defined property, that is, in a world where one economic agent owns an entire water system (river, lake, aquifer, etc.) the economic agent will maximize the welfare function over time. We appeal to government to act in this way when the Coase Theorem is thwarted by high transactions costs.

$$\max_{\{z(t)\}} W = \int_0^\infty [U(z(t)) + x - h(z(t))D(t)]e^{-rt} dt$$

Dynamic optimization involves choosing z_t so that it maximizes utility both now and in the future. The optimal choice maximizes the present value of welfare. The mathematical procedure is to recognize that the maximum for welfare, *W**, is defined in terms of an optimal path for resource use, *z**(*t*), that then implies optimal values for the resource stock, i.e., *D**(*t*). Welfare varies through time accordingly; call this *w**(*t*). At the maximum, a change in *D* will have a shadow price; call this λ(*t*). That is,

$$\frac{\partial w^*(z^*(t), D^*(t))}{\partial D} = \lambda(t) \qquad 1$$

This is the marginal cost of *D*; it is negative. It is the equivalent to the price of access to the resource. From this we can define the total discounted cost of resource access at each moment in time as:

$$\lambda(t)D(t)e^{-rt}$$

and the time path of changes in this is given by the time derivative:

$$\frac{\partial\, [\lambda(t)D(t)e^{-rt}]}{\partial t} = \dot{\lambda}De^{-rt} + \dot{D}\lambda e^{-rt} - r\lambda De^{-rt} \qquad 2$$

where $\dot{D} = f(z-R)$.

Welfare is maximized by maximizing the value of current production in each period and minimizing the cost of *D* in future periods caused by a change in current production. (Remember λ is negative in (1) so adding (2) to the welfare function is the same as subtracting and hence minimizing the cost of access.) This can be written as:

$$\max_{\{z(t), D(t)\}} w = [U(z(t)) + x - h(z)D + \dot{\lambda}D + f(z-R)\lambda - r\lambda D]e^{-rt}$$

The FOC are:

$$\frac{\partial w}{\partial z} = [U_z - h_z D + \lambda f']e^{-rt} = 0$$
$$\frac{\partial w}{\partial D} = [-h + \dot{\lambda} - r\lambda]e^{-rt} = 0$$

In the steady state, $\dot{\lambda} = 0$ and *z* = *R*, so the FOC become:

$$U_z(R) = h_z(R)D + \frac{h(R)f'(0)}{r} \qquad 3$$

Equation (3) can be rewritten as:

$$p = MC + \left[\frac{\frac{C^* f'}{D}}{r} \right] \qquad 4$$

The left-hand side is the current marginal benefit of increasing resource usage. The first term on the right-hand side is the current marginal cost of increasing resource usage. The second term is the marginal cost of reducing the stock of the resource. Increasing current usage decreases the stock now and in the future. Dividing by *r* gives us the present value of this future cost. Equation (4) expresses this last term as a cost per unit of *D* so that it is more intuitively of the same magnitude as price and marginal cost.

If the resource is owned, maximizing economic agents will obey this FOC and optimally use the resource. However, if the resource is not owned, then economic agents will only set current marginal benefits equal to current marginal costs. The equilibrium *D* when the resource is not owned is $D = U_z(R)/h_z(R)$ which is less than the equilibrium when the resource is owned, i.e.,

$$D = \frac{U_z(R)}{h_z(R)} - \frac{h(R) f'(0)}{r h_z(R)}$$

In the equilibrium, the consumption rate of *q* and its price is the same regardless of the property rights regime. The difference is the cost of extracting the resource. To see this it is informative to examine the time series of the variables as they approach the steady state. The variables of interest are *z*, *D*, *p* (which is U_z), and λ. In the common access case, *z* starts big and falls rapidly to *R*. In the property rights case, *z* starts smaller, crosses the time path of the common access *z*, and approaches *R* more gradually. *D* and *p* follow similar patterns between the two cases.

For illustration consider h_z to be constant. Marginal cost is $h_z D$ which starts out at $h_z D^0$. As *D* goes up, this MC shifts up. In the common access case, firms set MC equal to *p*. As the resource is depleted, MC shifts up. It reaches $p(q(R))$ which is $U_z(R)$ at the steady state. At this point, the entire amount of price times quantity is consumed in the cost of extracting *R*.

In the perfect property rights case, price is equal to MC plus $f' h_z z/r$ minus $f' \dot{\lambda}/r$. Note that $h_z z = h$ because h_z is constant. Let f' be constant as well. Also, $\dot{\lambda}$ is not a function of *z*, so it also acts as a shifter. The full MC facing the firm in the perfect property rights case is, then, a straight line with a positive slope. It starts out above the flat MC of the common access firm. However, at the steady state, it intersects the MC of the common access firm from below. The rectangle that is two times the triangle between the two MC's is the cost savings and efficiency of perfect property rights.