

Lecture 33.doc
General Equilibrium & Pareto Efficiency

Basic Set-up

Consumers maximize utility from n goods constrained by a budget constraint that includes some set of these goods as endowments. That is:

$$\max_{\{x_{ij}\}} U_j = U_j(x_{1j}, \dots, x_{nj}) \quad 1$$

subject to

$$\sum_i p_i(\bar{x}_{ij} - x_{ij}) = 0 \quad 2$$

where \bar{x}_{ij} are the endowments. The consumers must forego consumption of some of the amounts of the endowments in order to pay for purchases of other goods.

The FOC imply:

$$U_{ij} - \lambda p_i = 0$$

$$\sum_i p_i(\bar{x}_{ij} - x_{ij}) = 0$$

Let $p_n=1$, then the FOC become:

$$U_{ij} - U_{nj} p_i = 0, \quad i = \{1, n-1\} \quad 3$$

$$\sum_{i=1}^n p_i(\bar{x}_{ij} - x_{ij}) = 0 \quad 4$$

The market clearing condition is:

$$\sum_j x_{ij} = \sum_j \bar{x}_{ij}, \quad i = \{1, n\} \quad 5$$

In the equilibrium for the system, the variables are x_{ij} where $i=\{1, n\}$ and $j=\{1, L\}$, which is nL terms, and p_k where $k=\{1, n-1\}$. The total number of variables is then $nL + n-1$. The number of equations is $nL + n$. This is seen as follows: (3) holds for $n-1$ goods and all L consumers; (4) holds for all L consumers; and (5) holds for all n goods. The extra equation makes the system redundant.

This is resolved by Walras' Law. Summing (4) over all L individuals gives:

$$\sum_{j=1}^L \sum_{i=1}^n p_i(\bar{x}_{ij} - x_{ij}) = 0$$

which can be rewritten as

$$\sum_{i=1}^n p_i \sum_{j=1}^L p_i (\bar{x}_{ij} - x_{ij}) = 0$$

Note that if $n-1$ of the market clearing conditions hold, then the aggregated budget constraint for those $n-1$ goods also holds. This implies that the n^{th} market must definitionally clear. That is, if

$$\sum_j x_{ij} = \sum_j \bar{x}_{ij}, \quad i = \{1, n-1\}$$

then

$$\sum_{i=1}^{n-1} p_i \sum_{j=1}^L (\bar{x}_{ij} - x_{ij}) = 0$$

which means

$$\sum_{j=1}^L (\bar{x}_{nj} - x_{nj}) = 0.$$

The redundant equation is automatically satisfied when the other equations hold. The redundant equation is associated with the arbitrarily initialized price, p_n , which is set to one.

Pareto Optimality

Pareto optimality is defined as a state where no one can be made better off without making someone worse. This state can be defined by considering the maximization of an individual's utility subject to holding the utility of all other individuals constant.

That is,

$$\max U_j = U_j(x_{1j}, \dots, x_{nj})$$

subject to

$$U_k(x_{1j}, \dots, x_{nj}) = \bar{U}_k, \quad k = \{1, L\}, k \neq j.$$

The optimization is also constrained by the economy-wide production possibility frontier.

$$F(x_1, \dots, x_n) = 0$$

where $x_i = \sum_{j=1}^L x_{ij}$. This process must be undertaken for each person in the economy.

The Pareto optimality process can be succinctly summarized mathematically in terms of a Lagrangian function:

$$\max W = \sum_{j=1}^L \lambda_j [U_j(x_{1j}, \dots, x_{nj}) - \bar{U}_j] + \lambda_{L+1} F(x_1, \dots, x_n)$$

The function is maximized L times where each time a different $\lambda_j, j=1, L$, is set equal to 1.

In spite of the apparent complexity of this process, the FOC are all the same and very simple:

$$\frac{U_{sj}}{U_{rj}} = \frac{U_{sk}}{U_{rk}} = \frac{F_s}{F_r}$$

Market Efficiency

Let n include resources and finished goods. Let L stand for consumers and producers. Consumers maximize (1) subject to (2). The FOC look like:

$$\frac{U_{sj}}{U_{rj}} = \frac{p_s}{p_r} \quad 6$$

Firms maximize profits which can be defined as:

$$\pi_h = \sum_i p_i x_{ih}$$

subject to

$$f_h(x_{1h}, \dots, x_{nh})$$

where the goods, x_i , that are purchased have negative signs and the goods that are sold have positive signs. Firms have no initial endowments. The implicit production function $f_h(\cdot)$ maps inputs with implicitly negative signs into outputs.¹

The FOC look like:

$$\frac{\partial \pi_h}{\partial x_{rh}} = p_r + \mu_h f_{hr} = 0$$

which becomes

$$\frac{p_r}{p_s} = \frac{f_{hr}}{f_{hs}} \quad 7$$

This says that firms make marginal decisions between inputs and outputs, inputs and inputs, and outputs and outputs based on the pairwise relative prices.

In the economy as a whole, the production possibilities frontier, which is expressed as pairwise comparisons between inputs and outputs, is the tradeoff between one good and another. The same thing is true for the production function of the firms. Hence, we can write:

¹ It is useful to consider a simple example. Let f be a standard Cobb-Douglas production function with one output and two inputs. In this example the problem looks like a normal profit maximization set-up except that the accounting summation of revenues and costs has all positive signs. Even so, when we impose the rule that inputs have an implicitly negative sign, the FOC are identical to the normal problem.

$$\frac{F_r}{F_s} = -\frac{\partial x_s}{\partial x_r} = \frac{f_{hr}}{f_{hs}} \quad 8$$

Equation (8) implies that the market is efficient because from (6) consumers base marginal consumption decisions on relative prices and from (7) firms base marginal production decisions on these same relative prices and based on (8) the marginal production decisions of the firm are identical to the marginal social tradeoffs between goods. Implicitly this means that for the competitive market to be Pareto optimal, the private rate of transformation must be equal to the social rate of transformation. Factors that drive a wedge between private and social production tradeoffs destroy the efficiency of the market. Such situations include: public goods, common access resources, and externalities.