

■ A PROBLEM OF CONSTRAINED MAXIMIZATION¹

Consider the problem of keeping bees. Bees provide pollination services and they make honey. For decades economists used the example of bees as the case of a positive externality. Mostly it was treated as a heuristic as opposed to a real-world example, but some economists chose to show how the market solved this externality example in the real world. Steve Cheung showed how bee keepers sometimes paid and sometimes received pay depending on the marginal value of their pollination services relative to the value of the honey they collected.² In this lecture, we analyze the problem of keeping bees as an integrated endeavor. We do this primarily to discuss the general problem of constrained optimization and the methodology of comparative statics.

- Consider a farmer that uses bees to pollinate apple trees and to make honey from nectar. The farmer's outputs are apples and honey. Assume that there is a tradeoff between these two outputs. That is, the nectar gathered from apple trees does not produce as much honey as that found elsewhere. When the farmer is interested in apple pollination, he places the hives close to the apple trees. This reduces the amount of honey while increasing the production of apples. Conversely, when the farmer increases honey production, apple production is sacrificed. Let this production function be characterized by $y=f(x_1,x_2)$, where y is the number of bees, x_1 is honey, and x_2 is apples. This is an inverse production function since the input is the left-hand-side variable; the partial derivative of input with respect to one of the outputs, f_i , tells us how much the input must increase for a one unit change in the output holding the other output constant.

Assume that the farmer has a fixed number of bees. The farmer maximizes profits by allocating the bees based on the prices of honey, p_1 , and apples, p_2 . The problem is one of maximizing $\Pi = p_1x_1 + p_2x_2$ subject to the inverse production function holding the number of bees constant. The farmer has only one choice variable, that is, the placement of the fixed number of hives. However, we are interested in examining his behavior in terms of the amount of honey and apples that he brings to market. We analyze his behavior in terms of these two variables, knowing that the choice of one implies the choice of the other. Mechanically this relationship is identified by setting the optimization problem up using the method of Lagrange:

$$\max_{\{x_1, x_2, \lambda\}} \pi = p_1x_1 + p_2x_2 + \lambda(y - f(\cdot)) \quad 1$$

Notice in equation (1) that the objective function modified by the Lagrangian multiplier times the constraint is relabeled.

The FOC are:

$$\begin{aligned} p_i - \lambda f_i &= 0, \quad i = 1, 2 \\ y - f(\cdot) &= 0 \end{aligned} \quad 2$$

SSOC:

$$H = |\pi_{ij}| > 0, \quad \{i, j\} = \{1, 2, \lambda\} \quad 3$$

¹ Silberberg (1990) sections 6.3-6.6, 7.4

Layard and Walters (1978) section 6.3 p 189-192

² "The Fable of the Bees: An Economic Investigation," Journal of Law and Economics, (16) 1973, p. 11.

• Our method of analysis is to: Assume profit maximizing behavior on the part of the farmer in eqt. (1). Derive the behavioral implications in eqt's (2). Assume that eqt. (3) holds which means that the behavior described in eqt's (2) is profit maximization. This 1-2-3 process then **implies** that we can predict changes in behavior. That is, we can make statements about how behavior adjusts as the parameters in the problem change. This 1-2-3 process gives us the **Implicit Function Theorem**. From the implicit function theorem we can write:

$$x_i^* = x_i^*(p_j, y), \quad i, j = 1, 2 \quad 4$$

The implicit function theorem says that from eqt (2) the x_i^* are implicitly a function of the prices and the constraint, y . This means that $\partial x_i^* / \partial p_j$ and $\partial x_i^* / \partial y$ exist. These are the predictions of the model. These are the comparative statics of the model.

Now let us examine what happens to the production mix when we let the price of honey change. The comparative static analysis can be stated as follows:

$$\begin{bmatrix} -\lambda f_{11} & -\lambda f_{12} & -f_1 \\ -\lambda f_{21} & -\lambda f_{22} & -f_2 \\ -f_1 & -f_2 & 0 \end{bmatrix} \begin{bmatrix} \partial x_1^* / \partial p_1 \\ \partial x_2^* / \partial p_1 \\ \partial \lambda^* / \partial p_1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \quad 5$$

Solving by Cramer's rule,³

$$\frac{dx_1^*}{dp_1} = \frac{-H_{11}}{H} = \frac{f_2^2}{H} > 0 \quad 6$$

$$\frac{dx_2^*}{dp_1} = \frac{-H_{12}}{H} = -\frac{f_1 f_2}{H} < 0 \quad 7$$

which show the expected results. When the price of honey increases, honey production goes up and apple production falls.⁴

• In addition to the question of what happens to each product separately when the price of honey increases, we might ask whether honey production increases by more or less than apple production falls. The answer to this question is given by

$$\frac{dx_1^*}{dp_1} + \frac{dx_2^*}{dp_1} = \frac{f_2^2}{H} - \frac{f_2 f_1}{H} = \frac{f_2(f_2 - f_1)}{H} >> 0, \text{ as } \frac{f_2}{f_1} >> 1 \quad 8$$

³ The implicit function theorem says that we can solve by Cramer's rule. Because H is nonzero (in this case positive), the matrix $[h_{ij}]$ can be inverted. In other words, eqt. (5) would be undefined if H were zero.

⁴ This is worth emphasizing that while these results are almost trivially obvious, our efforts here are to develop skills to be applied to problems where the answers are not so obvious. The only way to become confident in our skills is to practice them where we already know the answer.

The term f_2/f_1 is the negative of the slope of the inverse production function holding the number of bees constant and examining the tradeoff between x_1 and x_2 . This is seen by taking the total differential of the production function and holding the number of hives constant: $dy = f_1 dx_1 + f_2 dx_2 = 0$. When total hives are held constant, the production function becomes an implicit function wherein x_1 can be written as a function of x_2 [i.e., $x_1 = x_1(x_2)$], and dx_1/dx_2 is the derivative of this implicit function.⁵ From the total differential, the value of this derivative is $-f_2 f_1^{-1}$. This implicit function, which can be called the "production possibilities frontier," is negatively sloped for positive inverse marginal products (the f_i 's).

The interpretation of eqt. (8) is best done graphically. We are tempted to simply assert that the production possibilities frontier is concave. However, to be more rigorous, we need to deduce the shape of the production frontier. That is, we need to know the slope of slope.

We start with the slope of the production frontier recognizing all of the arguments implied by the nature of the implicit functional relation between honey and apples:⁶

$$\frac{dx_1}{dx_2} = - f_2(x_1(x_2), x_2) f_1(x_1(x_2), x_2)^{-1} \quad 9$$

Differentiating w.r.t. each x_2 term in turn, the second derivative of the slope is

$$\frac{d^2 x_1}{dx_2^2} = - f_{21} \frac{dx_1}{dx_2} f_1^{-1} - f_{22} \frac{dx_2}{dx_2} f_1^{-1} + f_2 f_1^{-2} f_{11} \frac{dx_1}{dx_2} + f_2 f_1^{-2} f_{12} \frac{dx_2}{dx_2} \quad 10$$

Substituting from (9):

$$= + f_{21} f_2 f_1^{-1} f_1^{-1} - f_{22} f_1^{-1} - f_2 f_1^{-2} f_{11} f_2 f_1^{-1} + f_2 f_1^{-2} f_{12} \quad 11$$

Collecting terms:

$$= - f_{22} f_1^{-1} + 2 f_{21} f_2 f_1^{-2} - f_{11} f_2^2 f_1^{-3} \quad 12$$

Simplifying:

$$= - f_1^{-3} [f_{22} f_1^2 - 2 f_{21} f_2 f_1 + f_{11} f_2^2] \quad 13$$

If eqt. (13) is negative then the production frontier is concave; it is negative if the term in brackets is positive. We can sign the bracketed term in eqt. (13) by reference to the SSOC.

⁵ The use of the "d" notation emphasizes the fact that there is only one variable in this problem. Once x_2 is chosen, x_1 is simultaneously determined.

⁶ It should now be clear why Silberberg spends what seems to be an inordinate amount of time on implicit functions in the early chapters.

$$H = \begin{vmatrix} -\lambda f_{11} & -\lambda f_{12} & -f_1 \\ -\lambda f_{21} & -\lambda f_{22} & -f_2 \\ -f_1 & -f_2 & 0 \end{vmatrix} \quad 14$$

Expanding along the third column:

$$H = -f_1 f_2 \lambda f_{21} + f_1^2 \lambda f_{22} + f_2^2 \lambda f_{11} - f_1 f_2 \lambda f_{12} \quad 15$$

$$H = \lambda [f_1^2 f_{22} + f_2^2 f_{11} - 2 f_1 f_2 f_{12}] \quad 16$$

For a maximum, H must be positive. We will discuss the value of λ in a moment but suffice to say here that it is positive by the definitions of the model and the FOC, and since it is, then H is positive if and only if the bracketed term in eqt. (16) is positive. This term is the same as the term in brackets in eqt. (13). Therefore, the SSOC require that the second derivative of the slope of the production possibilities frontier be negative, which is the same thing as requiring that the production possibilities frontier be concave.

This means that we have the picture of a concave production frontier on which the farmer allocates bees between honey and apples such that the slope of the frontier at the production point is equal to the price ratio between the outputs. Since $\Pi = p_1 x_1 + p_2 x_2$, we can rewrite this price line as $x_1 = \Pi / p_1 - (p_2 / p_1) x_2$. As the price of honey increases the price line becomes flatter. If that slope is greater than $|1|$ in $\{x_1, x_2\}$ space then total output increases as the price of honey increases. However, after some point, the slope falls below one (in absolute value) and total output falls as p_1 increases.

- Comparative static analysis lets us determine how behavior changes as parameters change. Behavior is defined in terms of the choices of x_1 and x_2 . If we know how these choices vary, we can make statements about the effects on the objective function. Specifically, in this case, we can determine how profit changes as, say, the price of honey increases. To do this we solve the objective function at the solution values of the x 's and λ . We know that the optimized value of profits must logically be defined in terms of Π , and even if we express the value in terms of π as:

$$\pi^* = p_1 x_1^* + p_2 x_2^* + \lambda^* [y - f(x_1^*, x_2^*)] \quad 17$$

π^* must equal Π^* because the constraint is expressed as zero. Even so, to derive the comparative static results concerning the value of the objective function, we must operate on the Lagrangian-modified objective function because as we move from one optimum to another, we must move along the constraint. Hence, we differentiate π^* with respect to p_1 :

$$\frac{d\pi^*}{dp_1} = x_1^* + \frac{dx_1^*}{dp_1} (p_1 - \lambda f_1) + \frac{dx_2^*}{dp_1} (p_2 - \lambda f_2) + \frac{d\lambda^*}{dp_1} (y - f(\cdot)) = x_1^* > 0 \quad 18$$

Notice that p_1 shows up six times in eqt. (17). However, the derivative w.r.t. all but one of these cancel out. Everywhere p_1 is an argument in the functions defined by the FOC, its derivative

ends up being canceled out by the FOC because they are solved equal to zero at the optimum. This result is general and is called the **Envelope Theorem**. In this case the result says that profits always go up when one price increases (and the other stays constant).⁷

Now we are ready to discuss the value of λ . Differentiate eqt. (17) with respect to y . The parameter y shows up in six places in eqt. (17). However, by the Envelope Theorem it only remains where it stands alone:

$$\frac{d\pi^*}{dy} = \lambda^* \quad 19$$

At the profit maximizing solution, the optimized value of the lagrangian multiplier, λ^* , measures the change in profits that occurs when y increases. Mathematics cannot tell us the value of this. Economics says that this must be positive. That is, given that we assumed that the prices of honey and apples are both positive and that the inverse marginal products are both positive, then λ^* must be positive based on the FOC. This implies that bees are valuable because based on eqt. (19) if the farmer gets more bees, profits go up.

The mathematics ties everything together nicely. Starting from the definitions and assumptions of the model, i.e., positive output prices and positive marginal products, the calculus says that bees are valuable and says that the production frontier must be concave if the SSOC are satisfied. Given a concave production frontier, the behavioral functions implicitly identified by the FOC predict the farmer's behavior in response to changes in prices.

⁷ We might ask what happens if the price ratio changes. This would be an equal but opposite change in both prices. The answer is found by taking the total differential of maximized profits.