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The Problem with Dogs

Let's consider a problem of consumer behavior that is somewhat more narrowly focused and mundane than is usual in advanced theory classes. As a thought experiment that should not be overly taxing let's try to observe and predict the behavior of people in regard to their treatment of dogs. A substantial portion of the population owns dogs. The calculus of this consumption is a commonplace, so much so that we should be able to analyze this decision making at the intuitive level if not the technical. More importantly, addressing a fairly straightforward problem such as this should allow us to step back from the most formal aspects of consumer theory, such as utility functions, marginal utility, and preference functions, which sometimes seem so arcane, and take a fresh look. From a different vantage point, stripped of excess baggage, we will try to focus on the real empirical nature of predicting consumer behavior.

I. Dog Haters

There are at least two theories that we can assert to describe the actions of people who have dogs. That is, there are two kinds of dog people. Let's start with dog haters—like me. I don't really hate dogs, at least not too much. I tolerate dogs when I am forced to because of friendship or kinship. But they are just animals. I don’t think they feel pain like we do. Hence, they don't need special treatment.

My behavior can be modeled in a fairly straightforward way by saying that I am simply interested in minimizing the expenditure necessary to achieve a requisite level of nutrition for a dog. For instance, if I keep my brother Robert's dogs, I want to spend the smallest amount possible to keep them healthy. Call nutrition N. Let nutrition be a function of the products x_1 through x_n that might be fed to dogs. That is,

$$N = N(x_1,\ldots, x_n)$$

In describing the nature of this function, suffice to say that each product has a positive effect on nutrition, i.e., N_i is positive.

Each of the products has a price, p_i. Expenditure is the sum of the prices times the quantities purchased. The hypothesized optimization problem that forms the assumed behavior of dog haters like me can be written as:

$$\min_{\{x_i\}} E = \sum_{i=1}^{n} p_i x_i + \lambda [N^0 - N(x_1,\ldots,x_n)] \quad 1$$

The FOC are:

$$\begin{aligned} \partial E/\partial x_i &= p_i - \lambda N_i(.) = 0, \; i = 1,n \\ \partial E/\partial \lambda &= N^0 - N(.) = 0 \end{aligned} \quad 2$$

The SSOC are given by the (n+1)-square, bordered Hessian:

$$H = \begin{vmatrix} -\lambda \ N_{ij} & \vdots & -N_j \\ \cdots & & \cdots \\ -N_i & \vdots & 0 \end{vmatrix}, \ i \& j = 1, n \qquad 3$$

All principal minors must be negative.

By the Implicit Function Theorem, the SSOC ensure that the FOC can be solved to yield equations describing the optimal values of the choice variables in terms of the parameters of the problem, most notably, the prices of the products and the target level of nutrition. Specifically, to say that these behavioral equations exist is to say that their partial derivatives with respect to their arguments are defined. That is, the FOC and SSOC imply that the optimal values of the dog food products, the x_i*s, can be written as $x_i^*(p_1,..., p_n, N^o)$.[1]

[1] Because the FOC are implicit and general functions, we cannot solve the FOC directly to identify the behavioral equations. Instead we must use an alternative approach. Instead of solving the FOC directly and then taking the partial derivative of each equation with respect to a particular parameter, we differentiate the entire set of FOC equations, evaluated at the optimal values, with respect to the parameter of interest. Having done this, we can solve the resulting system of equations for the partial of the variable of interest with respect to the parameter of interest.

The solution of this system of equations takes the familiar form: a matrix of coefficients composed of the elements of the bordered Hessian determinant identifying the SSOC is multiplied by a column vector of comparative static partial derivatives. This matrix product is equal to the vector of constants formed by the elements of the differentiation of the FOC where the parameter of interest stands alone. For instance, if we ask how behavior changes when the price of the i^{th} dog food changes, we differentiate w.r.t. p_i which gives:

$$\begin{bmatrix} -\lambda N_{11} & \cdots & -\lambda N_{1i} & \cdots & -\lambda N_{1n} & -N_1 \\ \vdots & \ddots & \vdots & & \vdots & \vdots \\ -\lambda N_{i1} & \cdots & -\lambda N_{ii} & \cdots & -\lambda N_{in} & -N_i \\ \vdots & & \vdots & \ddots & \vdots & \vdots \\ -\lambda N_{n1} & \cdots & -\lambda N_{ni} & \cdots & -\lambda N_{nn} & -N_n \\ -N_1 & \cdots & -N_i & \cdots & -N_n & 0 \end{bmatrix} \begin{bmatrix} \partial x_1^* / \partial p_i \\ \vdots \\ \partial x_i^* / \partial p_i \\ \vdots \\ \partial x_n^* / \partial p_i \\ \partial \lambda^* / \partial p_i \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ -1 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

A singular result of thinking about the problem in this way is the fact that the existence of the comparative static results depends on the ability to invert the matrix of coefficients.

$$\begin{bmatrix} \partial x_1^* / \partial p_i \\ \vdots \\ \partial x_i^* / \partial p_i \\ \vdots \\ \partial x_n^* / \partial p_i \\ \partial \lambda^* / \partial p_i \end{bmatrix} = \begin{bmatrix} -\lambda N_{11} & \cdots & -\lambda N_{1i} & \cdots & -\lambda N_{1n} & -N_1 \\ \vdots & \ddots & \vdots & & \vdots & \vdots \\ -\lambda N_{i1} & \cdots & -\lambda N_{ii} & \cdots & -\lambda N_{in} & -N_i \\ \vdots & & \vdots & \ddots & \vdots & \vdots \\ -\lambda N_{n1} & \cdots & -\lambda N_{ni} & \cdots & -\lambda N_{nn} & -N_n \\ -N_1 & \cdots & -N_i & \cdots & -N_n & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \vdots \\ -1 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

This matrix can be inverted if and only if its determinant is non-zero. Using Cramer's Rule, the solution of the system of equations depends on dividing by the determinant of the matrix of coefficients. If this determinant is

The important result in this problem is the partial of x_i with respect to p_i. Simply enough, the dog food demand curves holding nutrition constant are downward sloping. Differentiating the FOC with respect to the ith price gives:[2]

$$\frac{\partial x_i^*}{\partial p_i} = (-1)\frac{H_{ii}}{H} \qquad 4$$

which is negative:

$$\frac{\partial x_i^*}{\partial p_i} < 0 \qquad 5$$

We can repeat this process for each dog food product, i.e., $i = 1,..., n$. This says that dog haters (that is, people attempting to minimize expenditure subject to a nutritional goal) exhibit a downward sloping demand for each product they buy. If the price of any one product goes up, holding the prices of all other products constant, then their purchases of that product fall.

It is important to recognize that we have derived this result without appeal to anything like a utility function except that we have assumed that this kind of person is not a dog lover. The person faces a technical nutrition production function. The demand curves are empirically valid in the sense that if we can measure the level of nutrition that the individual feeds the dogs, and account for this in our empirical estimates, we can measure the demand curve.

However, it is reasonable to worry that our empirical estimates of dog food demand are conditioned on the assumption that we are observing dog haters. Will dog lovers behave differently than dog haters? If they do, how can we tell the difference between dog haters and dog lovers? (Do dog haters kick their pups? What if they do it at night? How could we know?)

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zero, then the solution is undefined. This determinant is the Hessian defining the SSOC. When we assume that the SSOC are satisfied, we assume that a solution to the comparative static equations exists.

[2] Using Cramer's Rule, we can see that this derivative is minus one times the ratio of the border preserving principal minor of order (n-1) and the full bordered Hessian.

$$\frac{\partial x_i^*}{\partial p_i} = \frac{\begin{vmatrix} -\lambda N_{11} & \cdots & 0 & \cdots & -\lambda N_{1n} & -N_1 \\ \vdots & \ddots & \vdots & & \vdots & \vdots \\ -\lambda N_{i1} & \cdots & -1 & \cdots & -\lambda N_{in} & -N_i \\ \vdots & & \vdots & \ddots & \vdots & \vdots \\ -\lambda N_{n1} & \cdots & 0 & \cdots & -\lambda N_{nn} & -N_n \\ -N_1 & \cdots & 0 & \cdots & -N_n & 0 \end{vmatrix}}{\begin{vmatrix} -\lambda N_{11} & \cdots & -\lambda N_{1i} & \cdots & -\lambda N_{1n} & -N_1 \\ \vdots & \ddots & \vdots & & \vdots & \vdots \\ -\lambda N_{i1} & \cdots & -\lambda N_{ii} & \cdots & -\lambda N_{in} & -N_i \\ \vdots & & \vdots & \ddots & \vdots & \vdots \\ -\lambda N_{n1} & \cdots & -\lambda N_{ni} & \cdots & -\lambda N_{nn} & -N_n \\ -N_1 & \cdots & -N_i & \cdots & -N_n & 0 \end{vmatrix}}$$

In a minimization problem all border preserving principal minors (H_{ii} and H of particular interest here) must be of the same sign.

II. Dog Lovers

In order to get to the bottom of this quandary, let's analyze the behavior of the dog lover. While dog haters are interested in just keeping the dogs healthy, they are not interested in maximizing welfare. It is reasonable to say that the opposite is true of lovers. Dog lovers think that dogs are just like people—one of the family, so to speak. They want to keep their dogs fat and happy. An appropriate theoretical characterization of this behavior is to assume that the dog lover wishes to maximize canine nutrition, subject, of course, to some resource constraint. In other words, we define the problem of the dog lover to one of maximizing nutrition subject to an expenditure constraint.

$$\max_{\{x_i\}} N = N(x_1,\ldots,x_n) + \mu\left[E^0 - \sum_{i=1}^{n} p_i x_i\right] \qquad 6$$

The FOC are: 7

$$\partial N / \partial x_i = N_i(.) - \mu p_i = 0; i = 1,n$$

$$\partial N / \partial \mu = E^0 - \sum_{i=1}^{n} p_i x_i = 0$$

The SSOC are given by the (n+1)-square, bordered Hessian for this optimization problem:

$$H^E = \begin{vmatrix} N_{ij} & \vdots & -p_i \\ \cdots & & \cdots \\ -p_j & \vdots & 0 \end{vmatrix} \qquad 8$$

For a maximum, the border-preserving principal minors must alternate in sign, starting with positive for the (3 x 3) border-preserving principal minor.

By the Implicit Function Theorem, this problem also yields a set of behavioral equations for the *n* products. Let these be noted as

$$x_i^E = x_i^E(p_1,\ldots,p_n,E^0) \qquad 9$$

where the superscript *E* is used in place of the normal star notation. It designates the constraint in this optimization problem, i.e., the expenditure limit E^0.

Even though we have behavioral equations implied by the optimization model, in the case of dog lovers, they are not so clean in their comparative static implications. In the case of dog lovers, if we let the i^{th} price change, we have:

$$\begin{bmatrix} N_{ij} & \vdots & -p_i \\ \cdots & & \cdots \\ -p_j & \vdots & 0 \end{bmatrix} \begin{bmatrix} \partial x_k^E / \partial p_i \\ \cdots \\ \partial x_i^E / \partial p_i \\ \cdots \\ \partial \mu^E / \partial p_i \end{bmatrix} = \begin{bmatrix} 0 \\ \cdots \\ \mu \\ \cdots \\ x_i \end{bmatrix} \qquad 10$$

Solving by Cramer's rule gives:

$$\frac{\partial x_i^E}{\partial p_i} = \mu \frac{H_{ii}^E}{H^E} + x_i \frac{H_{n+1,i}^E}{H^E} \qquad 11$$

While the first term on the right-hand side of eqt. (11) can be signed by the SSOC,[3] the second term cannot be. Hence, we have no unambiguous prediction using only this analysis about the behavior of dog lovers when the price of the i^{th} kind of dog food goes up.

However, we are not quite at a stalemate.

III. Dog Hater Hires a Dog Lover[4]

Let's consider what happens if a dog hater hires a dog lover to care for his pets. For instance, suppose that my brother leaves his dogs with me once again only this time I don't feel like putting up with them so I hire Julie (my secretary) to take care of them for me. (Julie, of course, is a dog lover.) Since I have kept the dogs before, I know how much money it takes to keep them healthy, and I give precisely this amount of money to Julie to provide for their care. Living on a secretary's income, Julie is strapped for cash and cannot supplement the amount that I have provided. The question is, then, how does the dog lover in the employ of the dog hater behave?

One of the fundamental theorems of classical optimization theory tells us the answer. It is called the Duality Theorem. It says that any constrained optimization problem can be looked at as either a maximization or a minimization problem. The objective function in one problem becomes the constraint in the other. Moreover and more importantly, looked at either way the solution values in both problems are the same if the constraints are adjusted appropriately. That is exactly the case we are considering here. I only want to keep the dogs healthy. I minimize the expenditure necessary to do this. When I turn the dogs over to Julie, I give her a budget that equals the minimized expenditure that I would make on the dogs. Now, Julie wants to maximize the nutrition that is available for the dogs. However, faced with the budget constraint that I have imposed, she can do no better for the dogs than I would.

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[3] What is it?

[4] Silberberg (1990) section 4.6

Let's be precise. In the expenditure minimization problem, I chose optimal amounts of each kind of dog food and we called those $x_i*(p_1,..., p_n, N^o)$. From these we can express the optimally minimized expenditure level as $E^* = \sum_{i=1}^{n} p_i x_i^*$, which is the amount that I gave Julie. This expenditure level, which I found to be optimal, can also be written as $E^*(p_1,..., p_n, N^0)$. The Duality Theorem says that when Julie attempts to maximize nutrition but is faced with a budget constrained to be no more than $E*$, her optimized choices of dog food, $x_i^E = x_i^E(p_1,..., p_n, E*)$, will be identical to my x_i* choices. Spelled out we have

$$x_i^E(p_1,\ldots,p_n,E^*(p_1,\ldots,p_n,N^0)) \equiv x_i^*(p_1,\ldots,p_n,N^0) \qquad 12$$

From this identity, we can derive a very important result. Take the derivative of this equation with respect to p_i:

$$\frac{\partial x_i^E}{\partial p_i} + \frac{\partial x_i^E}{\partial E}\frac{\partial E^*}{\partial p_i} = \frac{\partial x_i^*}{\partial p_i} \qquad 13$$

Let's try to interpret eqt. (13). The left-hand side of (13) represents the effect of a change in price on the behavior of the dog lover. The right-hand side of (13) is the effect of a change in price on the dog hater. In this case, we know that the behavior of both the dog hater and the dog lover before the price change is identical and because of that, the change in their behavior when price changes is identical (at least in the region of the original optimum—called the LeChaterlier principle).

We know from equation (5) that the right-hand-side of equation (13) is negative. When the price of a given dog food goes up, dog haters buy less. Because of eqt. (13) we know that this means that dog lovers will respond similarly, all things considered. The beauty of eqt. (13) is that it tells us *all the things to consider*.

Eqt. (13) breaks the dog lover's response to a price change into two components: There is a price effect that is the first term on the left side of (13); and there is an income effect, which is the second, compound term. The sum of these two terms for the dog lover is equal to the simple price response by the dog hater. Think about what eqt. (13) says about Julie's behavior when the price of dog food goes up. The first term says that she responds to the price increase immediately; we expect[5] that this is negative, i.e., price goes up, quantity goes down. The second term implies a second step in Julie's behavior, and that step is an income response. Julie recognizes that the price increase has caused her purchasing power in feeding the dog to fall. In recognizing this, she comes to me for more money. This is the $\partial E*/\partial p_i$ term. I give her more money because I recognize that I would spend more money myself if I were taking care of the dogs. The extra money that I am willing to give her for the dogs times her own income effect forms the second part of Julie's reaction.

The sum of the two parts must, by the rules of calculus, be equal to my response, which we know obeys the Law of Demand.

[5] But cannot be sure.

IV. Dog Lovers act like Dog Haters[6]

What we have shown is that a dog lover constrained in budget terms by a dog hater will behave like the dog hater in terms of dog food price changes. However, the result is much more general. The behavior described in (13) is true for everyone, all the time. Let's consider my own brother's behavior (obviously, a dog lover). He maximizes nutrition in the same fashion as Julie. However, he is unconstrained by a budget from me and almost surely spends more than I do on the dogs. Even so, for whatever amount he does spend, eqt. (13) applies to him, his own self. Whatever nutrition level he achieves by his purchases can be modeled as the goal of an expenditure minimization process. True enough, when the price of a dog food goes up, Robert cannot come to me for more money for his dogs, but the component parts of his decision making are the same. There is a price effect and an income effect. In the normal case, the price effect means that he buys less of the particular dog food whose price increased and by the income effect (which is positive in the normal case) he is forced to spend more money on the dogs. In all cases, the magnitude of the combined price and income effects must be the negative response that he would reveal if he were targeting some nutrition level, because in the sense of the Duality Theorem he is.

Measurement is not a problem either. In Julie's case, when the price of a dog food goes up, she comes to me for more money. We don't see that when Robert is taking care of his own dogs. However, the Envelope Theorem helps us out here. From the Envelope Theorem, we know that $\partial E^*/\partial p_i$ is equal to x_i^* which is also equal to x_i^E.[7] So for Robert, (or me, or Julie, or anyone else) we have:

$$\frac{\partial x_i^E}{\partial p_i} + \frac{\partial x_i^E}{\partial E} x_i^E = \frac{\partial x_i^*}{\partial p_i} \qquad 14$$

The right-hand-side of equation (14) is the price effect that we observe when we hold nutrition constant. That is, if we examined data on purchases of dog food given information on prices and the nutrition level maintained by the dog keeper (lover or hater), the theory says that we will observe a downward sloping demand in product *i*. However, in general we may have a difficult time knowing the maintained level of nutrition. Even though the nutrition function, *N*(.), is a technical relation among the foods the dog eats, we will not normally be able to know all the things the dog eats. The dog keeper may let the dog forage or feed it table scraps. Without controlling for *N* we cannot observe equation (5) or the right side of (14).[8]

On the other hand, the left side of equation (14) tells us how the dog keeper behaves subject to a budget constraint. That is, the two expressions on the left-hand-side of equation (14) are the behavior responses we observe when we account for prices and budget. The first term is the change in the purchase of product *i* when the price of product *i* changes, holding budget constant. The second term is the change in purchases when the budget changes, holding price

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[6] Silberberg (1990) section 10.3 (review section 8.5-8.7 and section 10.2)

[7] This is a very important point. Do not let it breeze by. If you do not clearly understand this substitution, review the Envelope Theorem carefully.

[8] Difficult or not in principle, this very function has been estimated empirically. See Eugene Silberberg, "Nutrition and the Demand for Tastes," *JPE*, October 85, 881-900.

constant. These responses are generally observable. That is, the demand curves described by eqt. (9) require that we observe the dog keeper's choice of dog food *i*, the prices of all the different kinds of dog food, and the budget that the dog keeper spends on his dogs. In the absence of a budget for dogs, we could use overall income as a proxy.[9] Empirical estimates of eqt. (9) give us three different kinds of effects: the own price effect, $\partial x_i^E/\partial p_i$, the cross price effects, $\partial x_i^E/\partial p_j$, and the income effect, $\partial x_i^E/\partial E$. The first and last of these we need in evaluating eqt. (14).

It does not matter that we do not know whether the person is minimizing budget or maximizing nutrition. At the point of consumption, we know that equation (14) holds. Whichever objective the consumer is pursuing, duality implies the equivalence of the other in the neighborhood of the optimum. Equation (14) says that:

a) Conditioned on nutrition, the demand for product *i* is downward sloping.

b) Because of this, the slope of the budget-conditioned demand function for product *i* plus the weighted income effect given by this demand function must be negative.

Economics is an empirical science. Our efforts in predicting consumer behavior involve constructing a model. The value of a model of consumer behavior is measured in two related dimensions. First, we grade models on their ability to generate hard and fast predictions. Second, we want a model to organize a lot of information. The model of dog food consumption based on nutrition is a model that generates a hard and fast prediction. Holding nutrition constant, dog food demand is downward sloping. On the other hand, the dual of this model, even though it does not seem to generate the same hardness in predictions, assimilates a lot more information. While nutrition may be (probably is) a difficult datum to observe, budgeted expenditures are not. The model identified in budget terms makes it possible to predict the behavior of individual consumers based on observing their expenditures.

Also notice that the predictions we have derived are not based on a theory of utility. We have not concerned ourselves with preference functions nor any of their properties. We have merely assumed that people are somehow associated with dogs and, at a minimum, provide for them in a way that keeps them alive. In this regard, there is a well-behaved nutrition function that maps the technical relations among foods, but there is no worry about utility.

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[9] In fact, as we expand the model, dogs become an input in the utility function for consumers. We can model general consumer behavior in terms of the choice between dogs and everything else. The efficient expenditure on dog nutrition is then conditioned on overall income.