

**On Monopolies with Some Substitution Between Separated Markets:**

Assume the following demand functions characterize two markets served by a monopolist. Further assume that these demand curves reflect the ability of the monopolist to separate the markets and thereby price differently in each market.

$$q_1 = 1 - 2P_1 + P_2$$

$$q_2 = 1 + P_1 - P_2$$

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Let the cost of production be:  $C = \frac{q^2}{2} = \frac{(q_1 + q_2)^2}{2}$ .

There are two different ways to model this problem. The simplest is to substitute for quantities in the profit function and differentiate with respect to prices. Profit is:

$$\pi = P_1 q_1 + P_2 q_2 - \frac{(q_1 + q_2)^2}{2}$$

Substituting the demand expressions gives:

$$\max_{\{P_1, P_2\}} \pi = P_1 \cdot (1 - 2P_1 + P_2) + P_2 \cdot (1 + P_1 - P_2) - \frac{(1 - 2P_1 + P_2 + 1 + P_1 - P_2)^2}{2}$$

which simplifies to:

$$\max_{\{P_1, P_2\}} \pi = P_1 - 2P_1^2 + 2P_2 P_1 + P_2 - P_2^2 - \frac{(2 - P_1)^2}{2}$$

The FOC are

$$\frac{\partial \pi}{\partial P_1} = 1 - 4P_1 + 2P_2 + (2 - P_1) = 0$$

$$\frac{\partial \pi}{\partial P_2} = 2P_1 + 1 - 2P_2 = 0$$

which, in matrix form yields:

$$\begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} -3 \\ -1 \end{bmatrix}$$

The problem can alternatively be formulated in terms of quantities:

$$\max_{\{q_1, q_2\}} \pi = P_1(q_1, q_2)q_1 + P_2(q_1, q_2)q_2 - \frac{(q_1 + q_2)^2}{2}$$

This requires rewriting the demand curves. Expressing the demand curves in matrix form we can solve for the equilibrium price in each market as a function of the quantity level supplied to that market. That is,

$$\begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} q_1 - 1 \\ q_2 - 1 \end{bmatrix}$$

or

$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} q_1 - 1 \\ q_2 - 1 \end{bmatrix} \quad 2$$

The solution to the profit maximizing problem will be the same as when the choice variables of the firm are identified as the prices in the two markets.

Specifying the problem in quantities is somewhat more revealing. Notice what happens if the own-price effect in market one is equal to -1. In this event the matrix in (2) does not invert and the market prices are undefined. In other words if the own- and cross-price effects are equal both within and across markets, there is no equilibrium. The demand system does not exist.

More detail can be added if we generalize the problem. Rewrite the demand curves in (1) as:

$$\begin{aligned} q_1 &= a_0 + a_1 P_1 + a_2 P_2 + a_3 M \\ q_2 &= b_0 + b_1 P_1 + b_2 P_2 + b_3 M \end{aligned}$$

Solve for the joint equilibrium prices across the two markets:

$$\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} q_1 - a_0 - a_3 M \\ q_2 - b_0 - b_3 M \end{bmatrix}$$

Solving by Cramer's Rule gives:

$$P_1 = \frac{\begin{vmatrix} q_1 - a_0 - a_3 M & a_2 \\ q_2 - b_0 - b_3 M & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}} \quad 3$$

$$P_1 = \frac{\begin{vmatrix} a_1 & q_1 - a_0 - a_3 M \\ b_1 & q_2 - b_0 - b_3 M \end{vmatrix}}{\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}}$$

The determinant in the denominators of (3) is the product of the own-price effects minus the product of the cross price effects. Based on the homogeneity condition, the sum of the own and cross price effects each times their respective prices plus the income effect times income must equal zero. Hence, it is only if the income effect is zero in each market that the own and

cross price effects could be equal to one and opposite in sign thus canceling out. Our expectation is that the own-price effect will be larger (in absolute value) than the cross price effects and that the cross effects will be positive. Hence, the denominator will be positive.

Going a step further, recognize that the values  $a_0 + a_3M$  and  $b_0 + b_3M$  represent the consumption levels of  $q_1$  and  $q_2$ , respectively, that would be demanded if a zero price prevailed in both markets. The actual output supplied to each market will be less than this.<sup>1</sup> Hence,  $(q_1 - a_0 - a_3M)$  is negative, and the same is true in market two. This means that the numerators in the price equations defined by (3) are positive and thus both markets clear at positive prices given our other assumptions.

There are other possibilities for market equilibria but none are as clean as those described when the own-price effects are bigger than the cross-price effects and the cross effects are positive.

The most interesting aspect of the problem is to consider how to model substitution across the markets when there is some opportunity for buyers to arbitrage price. Is this captured in the normal cross-price effects in demand theory? Consider, for example, that I can eat at McD's in Clemson or I can drive to Seneca. I have a different demand for Big Mac's served in each place. Obviously, the price charged at one place affects my demand at the other. Is this the same thing that is going on in (1)?

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<sup>1</sup> In a competitive world for sure and most likely in a monopoly world unless something very strange is going on.